3/8/16 Lecture 19 outline

• Last time: QED at tree level has interaction vertex $-ie\gamma^\mu$ and propagators

$$D_{\mu\nu} = \frac{-i}{k^2}[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \xi\frac{k_\mu k_\nu}{k^2}], \qquad \frac{i}{\not k - m + i\epsilon},$$

(Popular choices: $\xi = 1$ is Feynman propagator, $\xi = 0$ is Landau gauge propagator. Physics is ξ independent (result of gauge invariance, which yields Ward-Takahashi identities). Let's choose to use Feynman gauge.) The photon has 1PI 2-point function $i\Pi^{\mu\nu}(k) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(k^2)$. Recall that the 1PI diagrams are defined with the external propagators amputated, and the full propagator is a geometric series: full = tree $\sum_{n=0}^\infty (1PI\cdot \text{tree})^n$. Writing it in Feynman gauge, the full propagator is $-ig_{\mu\nu}/p^2(1-\Pi(p^2))$. Assuming that $\Pi(p^2)$ is regular at $p^2 = 0$, get pole at $p^2 = 0$ with residue $Z_3 \equiv (1-\Pi(0))^{-1}$.

Likewise, for the electron propagator, defining the 1PI vertex to be $-i\Sigma(p)$, the electron has the full propagator $S(p) = i/(\not p - m - \Sigma(p))$, where for p near m, $S(p) = iZ_2/(\not p - m)$. The 1PI interaction vertex (with electron having incoming momentum p (and outgoing momentum $p+k$) and photon having incoming momentum k) is $-ie\Gamma^\mu(p+k,p)$, where for $k\to 0$, $\Gamma^\mu(p+k,p)\to Z_1^{-1}\gamma^\mu$.

• The W-T identity is

$$S(p+k)(-iek_\mu)\Gamma^\mu(p+k,p)S(p) = e(S(p) - S(p+k))$$

So

$$-ik_\mu\Gamma^\mu(p_k,p) = S^{-1}(p+k) - S^{-1}(p)$$

It's easily verified to work for the free propagators, and the W-T identity shows it's an exact result in the full, interacting theory. Taking p near on-shell and k near 0, this gives $Z_1 = Z_2$; this is an important consequence of gauge invariance. As we'll see more below, among other things, it ensures that e.g. the electron and the muon couple to the gauge field with the same effective charge.

• Compute the correction to the photon propagator from a virtual electron/positron loop:

$$i\Pi^{\mu\nu}(q) = -(-ie)^2\int\frac{d^4k}{(2\pi)^4}tr\left(\gamma^\mu\frac{i}{\not k - m}\gamma^\nu\frac{i}{\not k + \not q - m}\right).$$

Combine denominators using Feynman parameter

$$\frac{1}{(k^2-m^2)((k+q)^2-m^2} = \int_0^1 dx\frac{1}{(\ell^2 + x(1-x)q^2 - m^2)^2}$$

with $\ell = k + xq$. Go to Euclidean space and do integrals using our previous tables of integrals in dim-reg to find

$$\Pi(p^2) = -\frac{8e^2}{(4\pi)^{d/2}}\Gamma(2-\tfrac{1}{2}d)\int_0^1 dx x(1-x)\Delta^{\frac{1}{2}d-2},$$

with $\Delta = m^2 - x(1-x)p^2$. Evaluating for $d = 4-\epsilon$,

$$\Pi(p^2) = -\frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\left(\frac{2}{\epsilon} - \gamma + \log(4\pi/\Delta)\right).$$

We'll need to renormalize this.

• QED renormalization, similar to what we did in $\lambda\phi^4$. Bare and renormalized fields, and counterterms. $\psi_B = Z_2^{1/2}\psi_R$, $A_B^\mu = Z_3^{1/2}A_R^\mu$, $e_B Z_2 Z_3^{1/2} = e_R Z_1$. $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$.

$$\mathcal{L}_R = -\frac{1}{4}F_{R\mu\nu}F_R^{\mu\nu} + \bar{\psi}_R(i\partial\!\!\!/ - e_R A\!\!\!/_R - m_R)\psi_R,$$

$$\mathcal{L}_{ct} = -\frac{1}{4}\delta_3(F_{R\mu\nu})^2 + \bar{\psi}_R(i\delta_2\partial\!\!\!/ - \delta_1 e_R A\!\!\!/_R - \delta_m)\psi_R.$$

Where $\delta_1 = Z_1 - 1$, $\delta_2 = Z_2 - 1$, $\delta_3 = Z_3 - 1$, and $\delta_m = Z_2 m_0 - m$.

In particular, the counter-term contributes to $i\Pi^{\mu\nu}$ as $\delta\Pi = -(Z_3 - 1)$.

• Let's note some other interesting things about the finite part of $\Pi(p^2)$. $\Pi(p^2)$ has a branch cut starting at $p^2 = 4m^2$, and its imaginary part above and below the cut have

$$Im(\Pi(p^2 \pm i\epsilon) = \mp\frac{\alpha}{3}\sqrt{1-\frac{4m^2}{p^2}}(1+\frac{2m^2}{p^2}),$$

which is related by the optical theorem to the total cross section for creating an on-shell fermion-antifermion pair,

$$\frac{d\sigma}{d\Omega} = \frac{|\vec{p}|}{32\pi^2 s^{3/2}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2.$$