3/3/16 Lecture 18 outline

• Last time: Would like to integrate only over a slice of inequivalent gauge fields, without integrating over the gauge orbits. Need to do this, since otherwise there is no well defined B^{-1}. Recall $S = \int d^4x[-\frac{1}{4}F_{\mu\nu}^2] = \frac{1}{2}\int d^4k A_\mu(x)(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(x)$. Note action vanishes if $\tilde{A}_\mu(k) = k_\mu \alpha(k)$. Gauge invariance. $A_\mu^T = P_{\mu\nu}A^\nu$, $P_{\mu\nu} = g_{\mu\nu} - \partial_\mu\partial_\nu/\partial^2$. $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}A_\mu^T\partial^2 g^{\mu\nu}A_\nu^T$. Can't invert kinetic terms uniquely to find Green's function. We need to fix the gauge.

The functional integral should be over $\int[dA^\mu]/(GE)$, where we divide by the volume of the gauge equivalent orbits. The gauge equivalent orbits are associated with gauge transformations $\alpha(x)$, e.g. $A_\mu \to A_\mu + \partial_\mu\alpha(x)$ in the Abelian case. We want to do the functional integral over A^μ, dividing out by the $\alpha(x)$. Get

$$\int [dA]\Delta\delta(G[A])\exp(iS[A]).$$

where $G(A) = 0$ is some gauge fixing condition and Δ is the Faddeev-Popov determinant:

$$\Delta = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)_{G=0}.$$

• Take e.g. $G = \partial^\mu A_\mu - f(x)$ for some function $f(x)$. Then $\Delta \sim \det(\partial^2)$ is a constant. Get

$$e^{iW} = N\int(dA)e^{iS}\delta(\partial^\mu A_\mu - f) = N\int[dA][df]e^{iS}\delta(\partial^\mu A_\mu - f)G(f) = N\int[dA]e^{iS}G(\partial A),$$

for arbitrary functional G. Choose $G(f) = \exp(-\frac{1}{2}i\xi^{-1}\int d^4x f^2)$, for some real number ξ. Get

$$e^{iW} = N\int[dA]\exp(iS - \tfrac{1}{2}\xi^{-1}\int d^4x(\partial^\mu A_\mu)^2).$$

Then get for the propagator

$$D_{\mu\nu} = \frac{-i}{k^2}[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \xi\frac{k_\mu k_\nu}{k^2}].$$

Popular choices: $\xi = 1$ is Feynman propagator, $\xi = 0$ is Landau gauge propagator. Physics is ξ independent (result of gauge invariance, which yields Ward-Takahashi identities). Let's choose to use Feynman gauge.)

• Gauge invariance shows up in the amplitudes by what's know as the Ward-Takahashi identities. Consider a green's function $\langle 0|Tj^\mu(x)\prod_i\Phi(x_i)|0\rangle$, where j^μ is the conserved current and $\Phi(x_i)$ are other fields (they could be fermions). Much as you saw in a HW exercise, using the functional integral it is seen (by going through the symmetry transformation change of variables a-la Noether's procedure) that current conservation holds up to $\delta(x-x_i)$ contact terms. For example,

$$i\partial_\mu\langle 0|Tj^\mu(x)\psi(x_1)\overline{\psi}(x_2)|0\rangle = ie(\delta(x-x_2)-\delta(x-x_1))\langle 0|T\psi(x_1)\overline{\psi}(x_2)|0\rangle.$$

In momentum space,

$$-ik_\mu\mathcal{M}^\mu(k,p,q) = -ie\mathcal{M}_0(p,q-k)+ie\mathcal{M}_\prime(p+k,q).$$

Amplitudes with more external states are similar, with a sum over all external states weighted by their charge. When we go to S-matrix elements using the LSZ procedure, the terms on the RHS vanish when we amputate the external legs and go on-shell, so current conservation is indeed satisfied in S-matrix elements.

• Feynman rules for e.g. QED: propagator for free, spin 1/2 fermions:

$$\frac{i}{\not{k}-m+i\epsilon},$$

and gauge field

$$D_{\mu\nu} = \frac{-i}{k^2}[g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}+\xi\frac{k_\mu k_\nu}{k^2}]$$

Popular choices: $\xi=1$ is Feynman propagator, $\xi=0$ is Landau gauge propagator. Physics is ξ independent (result of gauge invariance, which yields Ward-Takahashi identities). Let's choose to use Feynman gauge.)

Recall QED Feynman rules, e.g. vertex: $-ie\gamma^\mu$.

• The photon has 1PI propagator $i\Pi^{\mu\nu}(k) = (p^2g^{\mu\nu}-p^\mu p^\nu)\Pi(k^2)$. Recall that the 1PI diagrams are defined with the external propagators amputated, and the full propagator is a geometric series: full = tree $\sum_{n=0}^\infty(1PI\cdot\text{tree})^n$. Writing it in Feynman gauge, the full propagator is $-ig_{\mu\nu}/p^2(1-\Pi(p^2))$. Assuming that $\Pi(p^2)$ is regular at $p^2=0$, get pole at $p^2=0$ with residue $Z_3\equiv(1-\Pi(0))^{-1}$.

Likewise, for the electron propagator, defining the 1PI vertex to be $\Sigma(p)$, the electron has the full propagator $S(p)=i/(\not{p}-m-\Sigma(p))$, where for p near m, $S(p)=iZ_2/(\not{p}-m)$. The 1PI interaction vertex (with electron having incoming momentum p (and outgoing momentum $p+k$) and photon having incoming momentum k) is $-ie\Gamma^\mu(p+k,p)$, where for $k\to 0$, $\Gamma^\mu(p+k,p)\to Z_1^{-1}\gamma^\mu$.