2/29/16 Lecture 16 outline

• Quantum field theory for fields with spin, in particular spin 1/2 fermions and spin 1 gauge fields, for example for QED [1]. Path integral of same, general form, but need to understand some new issues with the integrations.

Consider fermions first, where the functional integral is over grassmann valued fields. As you saw in a HW set, grassmann number integrals work like $\int d\theta(A+B\theta)=B$. Complex θ, θ^*, $\int d\theta^* d\theta \exp(-\theta^* b\theta) = b$. $\prod_i \int d\theta_i^* d\theta_i \exp(-\theta_i^* B_{ij}\theta_j) = \det B$. $\prod_i \int d\theta_i^* d\theta_i \exp(-\theta_i^* B_{ij}\theta_j)\theta_k\theta_l^* = (B^{-1})_{kl}\det B$.

• We can introduce sources for the fields:

$$\begin{aligned} Z[\bar\eta_i,\eta_i] &= \int d\bar\theta_i d\theta_i \exp(i(A_{ij}\bar\theta_i\theta_j + \bar\eta_i\theta_i + \bar\theta_i\eta_i]) \\ &= \int d\bar\theta_i d\theta_i (1 + i(\bar\theta, A\theta))(1+i\bar\eta\theta)(1+i\bar\theta\eta), \\ &= -i\det A \exp(-i\bar\eta_i A^{-1}_{ij}\eta_j). \end{aligned}$$

• Generalize to functional integrals over fermionic fields;

$$\begin{aligned} Z[\bar\eta,\eta] &= \int [d\bar\psi][d\psi]\exp(i\int d^4x[\bar\psi(i\partial\!\!\!/ - m)\psi + \bar\eta\psi + \bar\psi\eta] \\ &= Z_0 \exp[-\int d^4x d^4y \bar\eta(x) S_F(x-y)\eta(y). \end{aligned}$$

where

$$S_F[x-y] = i(i\partial\!\!\!/ - m)^{-1} = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(x-y)}}{k\!\!\!/ - m + i\epsilon}.$$

Get e.g.

$$\langle 0|T\psi(x)\bar\psi(y)|0\rangle = Z_0^{-1}(-i\frac{\delta}{\delta\bar\eta(x)})(i\frac{\delta}{\delta\eta(y)})Z[\eta,\bar\eta]|_{\eta,\bar\eta=0} = S_F(x-y).$$

This gives the Feynman rules for fermions that you saw last quarter.

• For fermions, the $\det B$ is in the numerator, whereas for scalars it's in the denominator. The functional integral gives e^{iW}. So the sign of the contribution to W is opposite for closed scalar vs fermion loops: every closed fermion loop gets an extra -1 factor. (This relative minus sign is put to good use with supersymmetry!)

[1] Fields with higher spin, e.g. the spin 2 metric, whose quanta are gravitons, can also be treated with the path integral, though they are non-renormalizable so a UV cutoff is required. Additional physics (e.g. string theory) can give a UV completion of the theory above the cutoff.

• Aside on anomalies in quantum field theory. The above was based on a real representation of fermions, in that both the Fermion and its complex conjugate were present. Can have Fermions that are in chiral representations, which is the case for all Fermions that we know (e.g. those in the Standard Model). Then we get the Pfaffian instead of the determinant, which is like the square-root of the determinant: for an antisymmetric matrix $B_{ij} = -B_{ji}$, with $i, j = 1 \dots 2N$, can form $PfB \sim B^{\wedge N}$ by contracting the indices with $\epsilon^{i_1 \dots i_{2N}}$.

• Recall spin 1 gauge field canonical quantization, and A_μ as an operator. Recall gauge invariance of $\mathcal{L}_{EM}$, needed to avoid having wrong sign kinetic term for longitudinal polarization terms.