2/23/16 Lecture 15 outline

• Last time,

$$\beta(\lambda) \equiv \frac{d}{d\ln\mu}\lambda_R$$

$$\gamma = \tfrac{1}{2}\frac{d}{d\ln\mu}\ln Z_\phi$$

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu}.$$

E.g. for $\lambda\phi^4$,

$$\beta(\lambda,\epsilon) = -\epsilon\lambda + \beta(\lambda)$$

$$\beta(\lambda) = \lambda^2\frac{da_1}{d\lambda}$$

$$\lambda^2\frac{da_{k+1}}{d\lambda} = \beta(\lambda)\frac{d}{d\lambda}(\lambda a_k),$$

$$\delta_m = \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_Z = 0.$$

So we find $a_1(\lambda) = +3\lambda/16\pi^2$ to one loop. This gives

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3).$$

• Likewise,

$$\gamma_\phi(\lambda,\epsilon) = \tfrac{1}{2}\frac{d}{d\ln\mu}\ln Z_\phi$$

where

$$Z_\phi = 1 + \sum_k Z_\phi^{-k}(\lambda)\epsilon^{-k}.$$

So

$$\gamma_\phi(\lambda,\epsilon) = \tfrac{1}{2}\beta(\lambda,\epsilon)\frac{d}{d\lambda}\ln Z_\phi.$$

Using $\beta(\lambda,\epsilon) = -\epsilon\lambda + \beta(\lambda)$, we get

$$\gamma_\phi = -\tfrac{1}{2}\lambda\frac{d}{d\lambda}Z_\phi^{(1)}.$$

We similarly have $m_B^2 = (m^2 + \delta_{m^2})Z_\phi^{-1} \equiv Z_m m^2$ and

$$\gamma_m(\lambda) = \tfrac{1}{2}\frac{d\ln m^2}{d\ln\mu} = -\tfrac{1}{2}\frac{d\ln Z_m}{d\ln\mu} = -\tfrac{1}{2}\beta\frac{d\ln Z_m}{d\lambda} = \tfrac{1}{2}\lambda\frac{dZ_m^{(1)}}{d\lambda}$$

where $Z_m^{(1)}$ means the coefficient of $1/\epsilon$. In all these cases, only the coefficient of $1/\epsilon$ matters. In particular, for $\lambda\phi^4$ we have

$$\gamma_m(\lambda) = \tfrac{1}{2}\lambda \frac{dZ_m^{(1)}}{d\lambda} = \tfrac{1}{2}\frac{\lambda}{16\pi^2} - \frac{5}{12}\frac{\lambda^2}{6(16\pi^2)^2} + \ldots$$

where $Z_m^{(1)}$ means the coefficient of $1/\epsilon$ and $\ldots$ are higher orders in perturbation theory, and

$$\gamma_\phi = -\tfrac{1}{2}\lambda \frac{d}{d\lambda} Z_\phi^{(1)} = \frac{1}{12}\frac{\lambda^2}{(16\pi^2)^2} + \ldots$$

For any gauge invariant field ϕ, we always have $\gamma_\phi \geq 0$, where $\gamma_\phi = 0$ iff it is a free field. This follows from the spectral decomposition result that $Z \leq 1$.

The anomalous dimension γ_ϕ is an additional quantum correction to the classical scaling dimension of the field: $\Delta(\mathcal{O}) = \Delta_{cl}(\mathcal{O}) + \gamma_{\mathcal{O}}$, e.g. here we find to 1-loop that $\Delta(\phi) = 1 + \frac{1}{12}\frac{\lambda^2}{(16\pi^2)^2}$.

• Let's discuss the RG equation in another way – the "Wilsonian" RG picture. Suppose that we break the path integral $\int [d\phi(k)]$ up into the "fast" modes, with $|k_E| > M$, and the "slow" ones with $|k_E| < M$, for some cutoff M. First do the integral over the fast modes, to get a low-energy effective theory lagrangian for the slow modes. This effective lagrangian has an effective coupling λ. Physics at the end of the day doesn't care about where we put M, but the effective coupling λ must vary with M to compensate for the fact that ultimately physics is M independent. Likewise, if we change $M \to M'$, we need to rescale $\phi' = Z_\phi^{-1/2}(M', M)\phi$. The condition that physics is independent of M is

$$\left(\frac{\partial}{\partial \log M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right)\widetilde{G}_R^{(n)}(p_1, \ldots p_n, M, \lambda) = 0,$$

with

$$\beta = \frac{d\lambda}{d\log M} \qquad \gamma = \tfrac{1}{2}\frac{d\ln Z_\phi}{d\ln M},$$

this is an alternative, equivalent interpretation of the same RG equations seen before (and we wrote it in terms of the un-amputated Green's function for variety).

• Integrating the 1-loop beta function that we found for $\lambda\phi^4$ theory gives

$$\lambda = \lambda_0\left(1 - \frac{3}{16\pi^3}\lambda_0 \ln(\mu/\mu_0)\right)^{-1}.$$

Or we can write the effective $\lambda(p) = \lambda(1 - (3\lambda/16\pi^2)\log(p/M))^{-1}$.

• All physical parameters, masses couplings etc satisfy the RG eqn:

$$\mathcal{D}P \equiv \left(\frac{\partial}{\partial \ln \mu} + \beta(\lambda)\frac{\partial}{\partial \lambda} + \gamma_m \frac{\partial}{\partial \ln m}\right) P(\lambda, m, \mu) = 0.$$

• Note: $\beta > 0$ means the coupling is small in the IR, and large in the UV. Such theories are "not asymptotically free" or are "IR free." Most theories are like this, e.g. $\lambda\phi^4$ (e.g. the Higgs coupling), QED, Yukawa interactions.

• QED: one loop beta function, $\beta(e) = e^3/12\pi^2$, leads to $\alpha_{eff}(\mu)^{-1} = \alpha_0^{-1} - \frac{1}{6\pi}\log(\mu/\mu_0)$. Again, positive beta function. Discuss the interpretation. Picture for QED of vacuum polarization, screening the bare charge.

4d theories without non-abelian gauge fields all have $\beta > 0$. They then need a cutoff to define them in the UV, and tend to flow to free theories in the IR.

• QCD has $\beta < 0$: the coupling is small in the UV, and large in the IR. Such theories are "asymptotically free;" only non-Abelian gauge theories, like QCD, are like that. Means vacuum anti-screens charges. QCD: one loop beta function $\beta(g) = -Cg^3/2$, leads to $g^{-2}(\mu) = g_0^{-2} + C\log(\mu/\mu_0)$.

• Pictures of RG flows. Briefly outline GUT idea and unification of the running couplings.

• New topic: quantum field theory for fields with spin, in particular spin 1/2 fermions and spin 1 gauge fields, for example for QED [1]. Path integral of same, general form, but need to understand some new issues with the integrations.

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[1] Fields with higher spin, e.g. the spin 2 metric, whose quanta are gravitons, can also be treated with the path integral, though they are non-renormalizable so a UV cutoff is required. Additional physics (e.g. string theory) can give a UV completion of the theory above the cutoff.