2/18/16 Lecture 14 outline

• Last time,

$$\tilde{\Gamma}^{(n)}_B(p_1, \ldots p_n; \lambda_B, m_B, \epsilon) = Z_\phi^{-n/2}\tilde{\Gamma}^{(n)}_R(p_1, \ldots p_n; \lambda_R, m_R, \mu, \epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point μ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with μ. Take $d/d\ln\mu$ of both sides, and use $d\Gamma_B/d\mu = 0$. This gives

$$\left(\frac{\partial}{\partial \ln \mu} + \beta(\lambda_R)\frac{\partial}{\partial \lambda_R} + \gamma_m \frac{\partial}{\partial \ln m_R} - n\gamma\right)\tilde{\Gamma}^{(n)}_R(p_1, \ldots p_n; \lambda_R, m_R, \mu) = 0$$

Here

$$\beta(\lambda) \equiv \frac{d}{d\ln\mu}\lambda_R$$

$$\gamma = \tfrac{1}{2}\frac{d}{d\ln\mu}\ln Z_\phi$$

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu}.$$

This is the RG equation. Various variants, depending on subtraction procedure (scheme). For mass dependent scheme, this gives the original Gell-Mann Low equations, where β and γ depend on the physical mass. The Callan-Symanzik equation replaces $\partial/\partial\ln\mu$ with $\partial/\partial\ln m$, giving the change as the physical mass is varied. It's often better to use a mass-independent scheme, like MS (or $\overline{MS}$, where we had introduced the scale M in replacing, via appropriate counterterms, $(\frac{2}{\epsilon} - \gamma + \log(4\pi/m^2) \to \log(M^2/m^2))$, where m appears as just another coupling constant. In any case, the RG equation can be integrated, to relate the renormalized Greens functions at different scales μ and μ'.

Recall also that $\tilde{\Gamma}^{(n)}$ are the 1PI diagrams with external propagators *amputated*. Alternatively, we could write RG equations for the Green's functions with the external propagators. Recall that $\tilde{G}^{(n)} \sim Z^n\tilde{\Gamma}^{(n)}$, which means that the RG equation differs from the above by $-n\gamma_\phi \to +n\gamma_\phi$; this is how you'll find it written in some texts.

• Physical picture in QED of the bare charge and running $\alpha(\mu)$.

• Understand what β and γ mean: the bare quantities are some function of the renormalized ones and epsilon. E.g. for $\lambda\phi^4$, recall $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$ and $\phi_B \equiv Z_\phi^{1/2}\phi_R$, so we had $\mathcal{L}_{c.t.} = \ldots - \delta\lambda\mu^\epsilon\phi^4/4!$ where $\delta\lambda\mu^\epsilon \equiv \lambda_B Z_\phi^2 - \lambda\mu^\epsilon$, which we'll rewrite as

$$\lambda_B = \mu^\epsilon Z_\phi^{-2}(\lambda + \delta_\lambda) \equiv \mu^\epsilon \lambda Z_\lambda$$

where

$$Z_\lambda \equiv Z_\phi^{-2}(1+\frac{\delta_\lambda}{\lambda}) \equiv 1+\sum_{k>0} a_k(\lambda)\epsilon^{-k}.$$

The bare parameter λ_B is independent of μ, whereas λ depends on μ, such that the above relation holds. Take $d/d\ln\mu$ of both sides,

$$0=(\epsilon\lambda+\beta(\lambda,\epsilon))Z_\lambda+\beta(\lambda,\epsilon)\lambda\frac{dZ_\lambda}{d\lambda}.$$

This equation must hold as a function of ϵ. Now $Z_\lambda = 1+\epsilon^{negative}$, and $dZ_\lambda/d\lambda = \epsilon^{negative}$. On the other hand, $\beta(\lambda,\epsilon) = d\lambda_R/d\ln\mu$ is non-singular as $\epsilon\to 0$, so $\beta(\lambda,\epsilon)=\beta(\lambda)+\sum_{n>0}\beta_n\epsilon^n$. Plugging back into the above equation then gives

$$\beta(\lambda,\epsilon) = -\epsilon\lambda+\beta(\lambda)$$

$$\beta(\lambda)=\lambda^2\frac{da_1}{d\lambda}$$

$$\lambda^2\frac{da_{k+1}}{d\lambda}=\beta(\lambda)\frac{d}{d\lambda}(\lambda a_k),$$

where the first comes from ϵ^n, the second from ϵ^0, and the third from ϵ^{-k}, with $n,k>0$.

So the beta function is determined entirely from a_1. The $a_{k>1}$ are also entirely determined by a_1. In k-th order in perturbation theory, the leading pole goes like $1/\epsilon^k$.

Recall that we found for $\lambda\phi^4$, in MS where we found to 1-loop

$$\delta_m=\frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon},\qquad \delta_\lambda=\frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon},\qquad \delta_Z=0.$$

So we find $a_1(\lambda)=+3\lambda/16\pi^2$ to one loop. This gives

$$\beta(\lambda)=\frac{3\lambda^2}{16\pi^2}+\mathcal{O}(\lambda^3).$$

- Likewise,

$$\gamma_\phi(\lambda,\epsilon)=\tfrac{1}{2}\frac{d}{d\ln\mu}\ln Z_\phi$$

where

$$Z_\phi = 1+\sum_k Z_\phi^{-k}(\lambda)\epsilon^{-k}.$$

So

$$\gamma_\phi(\lambda,\epsilon)=\tfrac{1}{2}\beta(\lambda,\epsilon)\frac{d}{d\lambda}\ln Z_\phi.$$

Using $\beta(\lambda, \epsilon) = -\epsilon\lambda + \beta(\lambda)$, we get

$$\gamma_\phi = -\tfrac{1}{2}\lambda \frac{d}{d\lambda} Z_\phi^{(1)}.$$

We similarly have $m_B^2 = (m^2 + \delta_{m^2}) Z_\phi^{-1} \equiv Z_m m^2$ and

$$\gamma_m(\lambda) = \tfrac{1}{2} \frac{d \ln m^2}{d \ln \mu} = -\tfrac{1}{2} \frac{d \ln Z_m}{d \ln \mu} = -\tfrac{1}{2}\beta \frac{d \ln Z_m}{d\lambda} = \tfrac{1}{2}\lambda \frac{dZ_m^{(1)}}{d\lambda}$$

where $Z_m^{(1)}$ means the coefficient of $1/\epsilon$. In all these cases, only the coefficient of $1/\epsilon$ matters.

In particular, for $\lambda\phi^4$ we have

$$\gamma_m(\lambda) = \tfrac{1}{2}\lambda \frac{dZ_m^{(1)}}{d\lambda} = \tfrac{1}{2} \frac{\lambda}{16\pi^2} - \frac{5}{12} \frac{\lambda^2}{6(16\pi^2)^2} + \ldots$$

where $Z_m^{(1)}$ means the coefficient of $1/\epsilon$ and $\ldots$ are higher orders in perturbation theory, and

$$\gamma_\phi = -\tfrac{1}{2}\lambda \frac{d}{d\lambda} Z_\phi^{(1)} = \frac{1}{12} \frac{\lambda^2}{(16\pi^2)^2} + \ldots$$