2/11/16 Lecture 12 outline

• Last time,

$$\frac{i}{p^2-m^2-\Pi'(p^2)+i\epsilon}=\frac{iZ}{p^2-m^2+i\epsilon}+\int_{\sim 4m^2}^{\infty}\frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2-M^2+i\epsilon}.$$

The LHS has a simple pole, with residue iZ, at $p^2=m^2$. Here $Z=|\langle\lambda_0|\phi(0)|\Omega\rangle|^2$ is the probability for $\phi(0)$ to create the lowest energy 1-particle state from the vacuum. Then there can be a few more simple poles, for p^2 slightly below $4m^2$.

Starting at $p^2=4m^2$, there is a branch cut, corresponding to producing two more more free particles. Note $\mathcal{A}(s)=\mathcal{A}(s^*)^*$ implies that the real part of $\mathcal{M}$ is continuous across the cut, but the imaginary part can be discontinuous: $Im\mathcal{A}(s+i\epsilon)=-Im\mathcal{A}(s-i\epsilon)$. We'll return to this shortly.

The above equality, back in position space and taking $\partial/\partial t$, leads to the equal time commutators, $[\phi(\vec{x},t),\dot{\phi}(\vec{y},t)]=i\delta^{(3)}(\vec{x}-\vec{y})$, matching the coefficient of the delta function on the two sides of the resulting equation gives

$$1=Z+\int_{\sim 4m^2}^{\infty}\frac{dM^2}{2\pi}\rho(M^2)\geq Z.$$

Implies that $0\leq Z\leq 1$, with $Z=1$ iff the theory is a free field theory. Intuitively reasonable, since Z essentially gives the probability of ϕ to create a 1-particle asymptotic in state, given that it can also create other things. Recall what we found before,

$$\delta_Z^{(2)}=-\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\epsilon},$$

so negative (for $\epsilon>0$).

• Recall LSZ (Lehmann, Symanzik, Zimmermann '55) from last quarter, now noting that there are Z factors: the S-matrix element for m incoming and n outgoing particles

$$\langle\mathbf{p_1}\ldots\mathbf{p_n}|S|\mathbf{k_1}\ldots\mathbf{k_m}\rangle=\lim_{o.s}\prod_{i=1}^{n}(p_i^2-m_i^2)Z_i^{-1/2}\prod_{j=1}^{m}(k_j^2-m_j^2)Z_j^{-1/2}\tilde{G}^{n+m}(-p_i,k_i).$$

Here $\tilde{G}^{n+m}$ is the full $n+m$ point Green's function, including disconnected diagrams etc. The limit is where we take the external particles on shell. In this limit, the $p_i^2-m_i^2$ and $k_j^2-m_j^2$ prefactors all go to zero. These zeros kill everything on the RHS except for the connected contributions to $\tilde{G}$. Accounting for the fact that we amputate the external propagators, which go like $iZ_i(p_i^2-m_i^2)^{-1}$, the above becomes

$$\langle\mathbf{p_1}\ldots\mathbf{p_n}|S|\mathbf{k_1}\ldots\mathbf{k_m}\rangle=Z^{(n+m)/2}\tilde{G}^{n+m}_{amp,conn,B}(-p_i,k_i)=\tilde{G}^{n+m}_{amp,conn,R}(-p_i,k_j)$$

Good: the physical S-matrix elements are computed from the renormalized Greens functions, which we take to be finite in our renormalization procedure.

More detail: Let $|k\rangle$ be the physical one-particle momentum plane wave state of the full interacting theory, normalized to $\langle k'|k\rangle = (2\pi)^3 2\omega_k \delta^{(3)}(\vec{k}'-\vec{k})$, and $\phi(x)$ the Heisenberg picture field. As discussed last time, the FT of $\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle \sim iZ/(p^2-m^2+i\epsilon)$ near $p^2=m^2$, so

$$\langle k|\phi(x)|\Omega\rangle = \langle k|e^{iP\cdot x}\phi(0)e^{-iP\cdot x}|\Omega\rangle = e^{ik\cdot x}\langle k|\phi(0)|\Omega\rangle \equiv e^{ik\cdot x}Z_\phi^{1/2}.$$

We scatter wave packets, with some profile $F(\vec{k})$, with F.T. $f(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}F(\vec{k})e^{-ik\cdot x}$, where we define $k_0 = \sqrt{\vec{k}^2+\mu^2}$, so $f(x)$ solves the KG equation. Now define

$$\phi^f(t) = iZ_\phi^{-1/2}\int d^3\vec{x}(\phi(\vec{x},t)\partial_0 f(\vec{x},t) - f(\vec{x},t)\partial_0\phi(\vec{x},t)).$$

This depends only on t, and we'll be interested in it at $t\to\pm\infty$, where it makes asymptotic **single-particle** in and out states: $\langle k|\phi^f(t)|\Omega\rangle = F(\vec{k})$ (the ∂_0's in $\phi^f(t)$ give a needed $2\omega_k$ to cancel that in $d^3k/(2\pi)^3 2\omega_k$), and $\langle n|\phi^f(t)|\Omega\rangle = \frac{\omega_{p_n}+p_n^0}{2\omega_{p_n}}F(\vec{p}_n)e^{-i(\omega_{p_n}-p_n^0)t}\langle n|\phi(0)|\Omega\rangle$, where $\omega_{p_n}\equiv\sqrt{\vec{p}_n^2+\mu^2}$, which has $\omega_{p_n}<p_n^0$ for any multiparticle state. So for **any** state ψ, $\lim_{t\to\pm\infty}\langle\psi|\phi^f(t)|\Omega\rangle = \langle\psi|f\rangle + 0$, where $|f\rangle \equiv \int\frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}F(\vec{k})|\vec{k}\rangle$, and the multiparticle states contributions sum to zero using the Riemann-Lebesgue lemma. Moreover, you can easily verify that (taking $f(|x|\to\infty)\to 0$)

$$iZ_\phi^{-1/2}\int d^4x f(x)(\partial^2+\mu^2)\phi(x) = \int dt\partial_0\phi^f(t) = (\lim_{t\to-\infty} - \lim_{t\to\infty})\phi^f(t).$$

This will be just what we wanted, to get our incoming and outgoing scattering states.

Make separated in states: $|f_n\rangle = \prod\phi^{f_n}(t_n)|\Omega\rangle$, and out states $\langle f_m| = \langle\Omega|\prod(\phi^{f_m})^\dagger(t_m)$, with $t_n\to-\infty$ and $t_m\to+\infty$. With some work, it can be shown that the $|_{-\infty}^{\infty}$ differences work out right so that

$$\langle f_m|S-1|f_n\rangle = Z_\phi^{-(n+m)/2}\int\prod_n d^4x_n f_n(x_n)\prod_m d^4x_m f_m(x_m)^*\prod_r i(\partial_r^2+m_r^2)G(x_n,x_m).$$

Take $f_i(x)\to e^{-ik_ix_i}$ at the end. Thus get that the S-matrix element for m incoming particles and n outgoing ones is given by

$$\langle\mathbf{p_1}\ldots\mathbf{p_n}|S|\mathbf{k_1}\ldots\mathbf{k_m}\rangle = Z_\phi^{-(n+m)/2}\lim_{o.s}\prod_{i=1}^n(p_i^2-m_i^2)\prod_{j=1}^m(k_j^2-m_j^2)\tilde{G}^{n+m}(-p_i,k_i).$$

Again, $\tilde{G}^{n+m}$ is the full $n+m$ point Green's function, including disconnected diagrams etc. The limit is where we take the external particles on shell. In this limit, the $p_i^2 - m_i^2$ and $k_j^2 - m_j^2$ prefactors all go to zero. These zeros kill everything on the RHS except for the connected contributions to $\tilde{G}$. Accounting for the fact that we amputate the external propagators, which go like $iZ_i(p_i^2 - m_i^2)^{-1}$, the above becomes

$$\langle \mathbf{p_1} \dots \mathbf{p_n} | S | \mathbf{k_1} \dots \mathbf{k_m} \rangle = Z^{(n+m)/2} \tilde{G}^{n+m}_{amp,conn,B}(-p_i, k_i) = \tilde{G}^{n+m}_{amp,conn,R}(-p_i, k_j)$$

Good: the physical S-matrix elements are computed from the renormalized Greens functions, which we take to be finite in our renormalization procedure.

• Write

$$-i\widetilde{\Delta}(p^2) = \frac{i}{p^2 - m^2 - \Pi'(p^2) + i\epsilon} = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim 4m^2}^{\infty} \frac{dM^2}{2\pi} \rho(M^2) \frac{i}{p^2 - M^2 + i\epsilon}.$$

So, using $\frac{1}{x \pm i\epsilon} = P(1/x) \mp i\pi\delta(x)$, argue that $\pi\rho(s) = 2Im\widetilde{\Delta}(s)$ for $s \geq 4m^2$. (The minus sign in the definition of $\widetilde{\Delta}$ above is related to the special definition of $\widetilde{\Gamma}^{(n)}$ for $n = 2$ and $\widetilde{\Delta} \sim 1/\widetilde{\Gamma}^{(2)}$.)

Analyticity properties. E.g. $2 \to 2$ scattering. $\mathcal{A}(s) = \mathcal{A}(s^*)^*$. Recall the definition

$$\langle f | (S - 1) | i \rangle \equiv i\mathcal{A}_{fi} (2\pi)^4 \delta^4(p_f - p_i)$$

sometimes write $\mathcal{M}$ instead of $\mathcal{A}$. E.g. for ϕ^4 at tree-level 2-2 scattering, get $\mathcal{A} \sim \lambda$.

The real part $Re\mathcal{A}$ is continuous across the real axis, whereas the Im part picks up a minus sign. So the discontinunity $Disc\mathcal{A}(s) = 2iIm\mathcal{A}(s + i\epsilon)$. E.g. $\frac{1}{x \pm i\epsilon} = P(1/x) \mp i\pi\delta(x)$ shows that the discontinunity of $\frac{1}{p^2 - m^2 + i\epsilon} - (c.c.)$ is $-2\pi i\delta(p^2 - m^2)$.

• Optical theorem. The S-matrix $S = U(t_f = \infty, t_i = -\infty)$ is unitary, $S^\dagger S = 1$. Write $S = 1 + iT$, then get $2Im(T) \equiv -i(T - T^\dagger) = T^\dagger T$.