2/9/16 Lecture 11 outline

- Last time, consider $\lambda\phi^4$ in MS. To one loop, we found

$$\delta_m = \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_Z = 0.$$

Now consider the propagator to two loops. Diagram 1 is a one-loop diagram with the 1-loop $\delta\lambda$ counterterm at the vertex. Diagram 2 is a one-loop diagram with the 1-loop δ_m counterterm on the internal propagator. Diagram 3 is a two-loop diagram which looks like a double-scoop of the 1-loop diagrams. Diagram 4 is a line which cuts through a circle (see your HW). Diagram 5 has no loops, but an insertion of the 2-loop δ_m and δ_Z counter terms. Let's consider the pole terms in the diagrams. Diagram 1 requires no new computation: we can obtain it from the previous 1-loop contribution to $-i\Pi'$ by simply replacing there $\lambda \to \delta\lambda$. This gives

$$-i\Pi'_{diag\ 1} = i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{3}{2}\left(\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2} + \frac{1}{\epsilon} - \frac{\gamma}{\epsilon}\right) + \mathcal{O}(\epsilon^0)$$

Diagram 2 has 2 propagators in the loop, with the 1-loop $-i\delta_m$ vertex insertion, which gives (using the integral given at the start, now with $n = 2$ instead of $n = 1$):

$$-i\Pi'_{diag\ 2} = i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{1}{2}\left(\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2} - \frac{\gamma}{\epsilon}\right) + \mathcal{O}(\epsilon^0)$$

where the overall $\frac{1}{2}$ is a symmetry factor, as in the 1-loop diagram. Diagram 3 contributes (with two symmetry factors of $\frac{1}{2}$)

$$-i\Pi'_{diag\ 3} = \frac{1}{4}(-i\lambda)^2\mu^{2\epsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{i}{k^2-m^2}\int\frac{d^Dq}{(2\pi)^D}\left(\frac{i}{q^2-m^2}\right)^2,$$

where q is the integral over the lower loop, which has two propagators. This gives

$$-i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{1}{2}\left(\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\ln\frac{m^2}{4\pi\mu^2} + \frac{1}{\epsilon} - \frac{2\gamma}{\epsilon}\right) + \mathcal{O}(\epsilon^0)$$

Diagram 4 gives

$$i\frac{\lambda^2}{(16\pi^2)^2}\left(-\frac{m^2}{\epsilon^2} + \frac{1}{\epsilon}\left(m^2\ln\frac{m^2}{4\pi\mu^2} + \frac{1}{12}p^2 + (\gamma - \frac{3}{2}m^2)\right)\right) + \mathcal{O}(\epsilon^0).$$

(The finite (ϵ^0) contribution to diagram 4 can be evaluated by writing out the integrals and using the Feynman trick, but it is quite complicated for general $m \neq 0$. In the HW, you will evaluate it for $m = 0$, where it simplifies.)

Diagram 5 are the two-loop counterterms, $i\delta_Z^{(2)}p^2 - i\delta_m^{(2)}$. We should then take for the 2-loop contributions to the counterterms

$$\delta m^{(2)} = \frac{\lambda^2}{(16\pi^2)^2}\left(\frac{2}{\epsilon^2} - \frac{1}{2\epsilon}\right)m^2,$$

$$\delta_Z^{(2)} = -\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\epsilon}.$$

The terms involving $\ln m^2/4\pi\mu^2$ all cancel. This happens for all loops. MS is a mass independent scheme, in that $\delta\lambda$, δZ, and $\delta m/m^2$ are independent of m and μ.

• Renormalized and bare Greens functions. Recall that $\Phi_B \equiv Z_\phi^{1/2}\phi$, and the definition of the 1PI Green's functions $\widetilde{\Gamma}^{(n)}$, and in particular that they have all n external propagators amputated. It then follows that

$$\tilde{\Gamma}_B^{(n)}(p_1, \ldots p_n; \lambda_B, m_B, \epsilon) = Z_\phi^{-n/2}\tilde{\Gamma}_R^{(n)}(p_1, \ldots p_n; \lambda_R, m_R, \mu, \epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point μ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with μ. Rewrite above as

$$Z_\phi^{n/2}\tilde{\Gamma}_B^{(n)}(p_1, \ldots p_n; \lambda_B, m_B, \epsilon) = \tilde{\Gamma}_R^{(n)}(p_1, \ldots p_n; \lambda_R, m_R, \mu, \epsilon).$$

Now the RHS is finite, so the LHS must be too. So we can take $\epsilon \to 0$ without a problem.

• Before getting into the renormalization group, let's take a little detour. Recall that

$$\int d^4x e^{ipx}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle = \frac{i}{p^2 - m^2 - \Pi'(p^2) + i\epsilon}.$$

Here $|\Omega\rangle$ is the full, interacting vacuum and ϕ are the full (Heisenberg picture) operators. Now insert a complete set of states (all single and multi-particle states) ,

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \prod_i \int \frac{d^3p_i}{(2\pi)^3}\frac{1}{2E_p(\lambda)}|\lambda_p\rangle\langle\lambda_p|$$

where λ are all eigenstates of the full H, and λ_p is a boosted version, to give an eigenstate of $\vec{P}$, with spatial momentum $\vec{p}$. Now use $\phi(x) = e^{iP_\mu x^\mu}\phi(0)e^{-iP_\mu x^\mu}$, so

$\langle\Omega|\phi(x)|\lambda_p\rangle = \langle\Omega|\phi(0)|\lambda_0\rangle e^{-ipx}$ (where $p^0 = E_p \equiv \sqrt{|\vec{p}|^2 + m_\lambda^2}$) and replace $\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \to \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m_\lambda^2 + i\epsilon}$ to get

$$\langle\Omega|\phi(x)\phi(0)|\Omega\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m_\lambda^2 + i\epsilon} e^{-ipx}|\langle\Omega|\phi(0)|\lambda_0\rangle|^2.$$

So

$$\int d^4x e^{ipx}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon},$$

where

$$\rho(M^2) = \sum_\lambda 2\pi\delta(M^2 - m_\lambda^2)|\langle\Omega|\phi(0)|\lambda\rangle|^2 > 0$$

is the Kallen-Lehmann spectral density. Find $\rho(M^2) = 2\pi\delta(M^2 - m^2)Z$ for $M^2 \ll 4m^2$. For M^2 slightly below $4m^2$ there are new delta functions, at the bound states. Starting at $4m^2$, $\rho(M^2)$ is some positive function. This implies that

$$\frac{i}{p^2 - m^2 - \Pi'(p^2) + i\epsilon} = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim 4m^2}^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon}.$$

The LHS has a simple pole, with residue iZ, at $p^2 = m^2$. Here $Z = |\langle\lambda_0|\phi(0)|\Omega\rangle|^2$ is the probability for $\phi(0)$ to create the lowest energy 1-particle state from the vacuum. Then there can be a few more simple poles, for p^2 slightly below $4m^2$.

Starting at $p^2 = 4m^2$, there is a branch cut, corresponding to producing two more more free particles. Note $\mathcal{M}(s) = \mathcal{M}(s^*)^*$ implies that the real part of $\mathcal{M}$ is continuous across the cut, but the imaginary part can be discontinuous: $Im\mathcal{M}(s + i\epsilon) = -Im\mathcal{M}(s - i\epsilon)$. We'll return to this shortly.

The above equality, back in position space and taking $\partial/\partial t$, leads to the equal time commutators, $[\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y})$, matching the coefficient of the delta function on the two sides of the resulting equation gives

$$1 = Z + \int_{\sim 4m^2}^\infty \frac{dM^2}{2\pi}\rho(M^2) \geq Z.$$

Implies that $0 \leq Z \leq 1$, with $Z = 1$ iff the theory is a free field theory. Intuitively reasonable, since Z essentially gives the probability of ϕ to create a 1-particle asymptotic in state, given that it can also create other things. Recall what we found before,

$$\delta_Z^{(2)} = -\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\epsilon},$$

so negative (for $\epsilon > 0$).