

1/5/16 Lecture outline

• Last quarter, we learned canonical quantization, Dyson's formulae, and Feynman diagrams. BTW, we are very fortunate to now have Dyson here at UCSD. In addition to putting Feynman's intuitions on firm footing in the early days of QFT, he has done so many other things in his remarkably broad and long lifetime as a highly original, visionary thinker. He is also very fun and approachable; you are encouraged to meet him.

In QM classes, the usual descriptions (Schrodinger and Heisenberg) are based on using the Hamiltonian, since that's the generator of time translations. Also, in our canonical quantization last quarter, time plays a special role – we had equal time commutation relations, time ordered products, etc. Feynman found a completely new description of QM, by intuitively thinking about double slit interference and realizing that empty space can be thought of as being filled with screens that are full of holes, so such interference and taking multiple paths is always there. The path integral generalizes immediately from QM to QFT, and for different types of fields. Unlike canonical quantization, it makes Lorentz and Poincare symmetry manifest, and also gives a way to define QFT beyond perturbation theory. The classical limit is clarified, as the stationary phase limit of an integral. Similar statements apply in optics. The path integral also helps to connect QFT with statistical physics, with the path integral analogous to the partition function.

Recall from last quarter. The field $\phi(x)$ is analogous to $q(t)$ in QM (indeed, QM is a particular case of QFT in one dimension), and its conjugate momentum is $\Pi = \partial\mathcal{L}/\partial\dot{\phi}$. These are operators, with equal time commutators

$$[\phi(t, \vec{x}), \Pi(t, \vec{x}')] = i\hbar\delta^3(\vec{x} - \vec{x}').$$

We'll usually set $\hbar = 1$. The S-matrix elements, used to compute scattering cross sections and lifetimes etc. are computed from an amplitude $\langle f|S|i\rangle$ which is related to the vacuum expectation values of time-ordered products of the fields. This is seen from Dyson's formula or from the LSZ derivation, to be discussed in more detail later:

$$\begin{aligned} \langle f|i\rangle &= \langle k_{1'} \dots k_{n'} | k_1 \dots k_n \rangle \\ &= i^{n+n'} \prod_{j'=1}^{n'} \int d^4x'_j e^{ik'_j x'_j} (\partial_{j'}^2 + m^2) \prod_{j=1}^n e^{-ik_j x_j} (\partial_j^2 + m^2) G_{n+n'}(x_1 \dots x_n, x_{1'} \dots x_{n'}), \end{aligned} \tag{1}$$

where

$$G_{n+n'}(x_1 \dots x_n, x_{1'} \dots x_{n'}) \equiv \langle 0|T\phi(x_{1'}) \dots \phi(x_{n'})\phi(x_1) \dots \phi(x_n)|0\rangle. \tag{2}$$

Using Wick's theorem,

$$T(\phi_1 \dots \phi_n) =: \phi_1 \dots \phi_n : + : \text{all contractions},$$

gives Feynman's rules for computing amplitudes, from Feynman diagrams.

- Our first topic will be to get an alternative derivation of the Feynman rules, using the Feynman path integral. This gives an alternative to canonical quantization for quantizing particles and fields, and additional insights into the Feynman diagrams and rules. First consider particle quantum mechanics (QFT in $d = 1 + 0$ spacetime dimensions). The probability amplitude to go from position q at time t to q' at time t' is $\langle q', t' | q, t \rangle$. Let's write this as the time evolution operator

$$U(x_a, x_b; T) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle.$$

Satisfies SE

$$i\hbar\partial_T U = HU.$$

Feynman:

$$U(x_a, x_b; T) = \int [dx(t)] e^{iS[x(t)]/\hbar}.$$

Integral can be broken into time slices, as way to define it.

E.g. free particle

$$\left(\frac{-im}{2\pi\hbar\epsilon}\right)^{N/2} \int \prod_{i=1}^{N-1} dx_i \exp\left[\frac{im}{2\hbar\epsilon} \sum_{i=1}^N (x_i - x_{i-1})^2\right]$$

Where we take $\epsilon \rightarrow 0$ and $N \rightarrow \infty$, with $N\epsilon = T$ held fixed.

Do integral in steps. Apply expression for real gaussian integral (valid: analytic continuation):

$$\int_{-\infty}^{\infty} d\phi \exp(ia\phi^2) = \sqrt{\frac{i\pi}{a}}.$$

where we analytically continued from the case of an ordinary gaussian integral. Think of a as being complex. Then the integral converges for $\text{Im}(a) > 0$, since then it's damped. To justify the above, for real a , we need the integral to be slightly damped, not just purely oscillating. To get this, take $a \rightarrow a + i\epsilon$, with $\epsilon > 0$, and then take $\epsilon \rightarrow 0^+$. We'll see that this is related to the $i\epsilon$ that we saw last quarter in the Feynman propagator, which gave the T ordering.

After $n - 1$ steps, get integral:

$$\left(\frac{2\pi i \hbar n \epsilon}{m}\right)^{-1/2} \exp\left[\frac{m}{2\pi i \hbar n \epsilon}(x_n - x_0)^2\right].$$

So the final answer is

$$U(x_b, x_a; T) = \left(\frac{2\pi i \hbar T}{m}\right)^{-1/2} \exp[im(x_b - x_a)^2/2\hbar T].$$

Note that the exponent is $e^{iS_{cl}/\hbar}$, where S_{cl} is the classical action for the classical path with these boundary conditions. (More generally, get a similar factor of $e^{iS_{cl}/\hbar}$ for interacting theories, from evaluating path integral using stationary phase.)

Plot phase of U as a function of $x = x_b - x_a$, fixed T , Lots of oscillates. For large x , nearly constant wavelength λ , with

$$2\pi = \frac{m(x + \lambda)^2}{2\hbar T} - \frac{2x^2}{2\hbar T} \approx \frac{m\lambda x}{\hbar T} = p\lambda/\hbar.$$

Gives $p = \hbar k$.

Recover $\psi \sim e^{ipx/\hbar}$. More generally, get $p = \hbar^{-1}k$, with $p = \partial S_{cl}/\partial x_b$ (can show $p = \partial L/\partial \dot{x} = \partial S_{cl}/\partial x_b$). Can also recover $\psi \sim e^{-i\omega T}$, with $\omega = \hbar^{-1}(-\partial S_{cl}/\partial t_b)$. Agrees with $E = \hbar\omega$, since $E = p\dot{x} - L = -\partial S_{cl}/\partial t_b$.

• Nice application: Aharonov-Bohm. Recall $L = \frac{1}{2}m\dot{\vec{x}}^2 + q\dot{\vec{x}} \cdot \vec{A} - q\phi$. Solenoid with $B \neq 0$ inside, and $B = 0$ outside. Phase difference in wavefunctions is

$$e^{i\Delta S/\hbar} = e^{iq \oint \vec{A} \cdot d\vec{x}/\hbar} = e^{iq\Phi/\hbar}.$$

Aside on Dirac quantization for magnetic monopoles.

• Can derive the path integral from standard QM formulae, with operators, by introducing the time slices and a complete set of q and p eigenstates at each step.

$$\langle q', t' | q, t \rangle = \int \int \prod_{j=1}^N dq_j \langle q' | e^{-iH\delta t} | q_{N-1} \rangle \langle q_{N-1} | e^{-iH\delta t} | q_{N-2} \rangle \dots \langle q_1 | e^{-iH\delta t} | q \rangle,$$

where we'll take $N \rightarrow \infty$ and $\delta t \rightarrow 0$, holding $t' - t \equiv N\delta t$ fixed. Note that even though $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B] + \dots}$, we're not going to have to worry about this for $\delta t \rightarrow 0$: $e^{-iH\delta t} = e^{-i\delta t p^2/2m} e^{-i\delta t V(q)} e^{\mathcal{O}(\delta t^2)}$. Now note

$$\langle q_2 | e^{-iH\delta t} | q_1 \rangle = \int dp_1 \langle q_2 | e^{-i\delta t p^2/2m} | p_1 \rangle \langle p_1 | e^{-iV(q)\delta t} | q_1 \rangle,$$

$$= \int dp_1 e^{-iH(p_1, q_1)\delta t} e^{ip_1(q_2 - q_1)}.$$

This leads to

$$\langle q', t' | q, t \rangle = \int [dq(t)][dp(t)] \exp(i \int_t^{t'} dt (p(t)\dot{q}(t) - H(p, q))),$$

and taking H quadratic in momentum and doing the p gaussian integral recovers the Feynman path integral.

- The same derivation as above leads to e.g.

$$\langle q_4, t_4 | T \hat{q}(t_3) \hat{q}(t_2) | q_1, t_1 \rangle = \int [dq(t)] q(t_3) q(t_2) e^{iS/\hbar},$$

where the integral is over all paths, with endpoints at (q_1, t_1) and (q_4, t_4) .

A key point: the functional integral automatically accounts for time ordering! Note that the LHS above involves time ordered operators, while the RHS has a functional integral, which does not involve operators (so there is no time ordering). The fact that the time ordering comes out on the LHS is wonderful, since know that we'll need to have the time ordering for using Dyson's formula, or the LSZ formula, to compute quantum field theory amplitudes.