1/15/16 Homework 2. Due Jan 21

1. Compute the 4-point function $\langle T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\rangle$ to order λ in $\lambda\phi^4$ scalar field theory by using the generating functional $Z[J]$. Connect the results of taking the $\delta/\delta J$'s with the diagrammatic notation, being careful with the coefficients. Show that this gives the Feynman rules that you know, e.g. one diagram is the 4-point vertex, weighted by $-i\lambda$. Verify that there are also disconnected contributions which involve one regular propagator, and one propagator with a loop correction, where the latter is the $O(\lambda)$ correction to the 2-point functions (like $\langle T\phi(x_1)\phi(x_2)\rangle$). Finally, there is the bubble diagram contribution, which cancels in the end (from the $1/Z[J]$ in our rules for using the generating functional). You don't need to evaluate the actual loop integrals for this problem, the point is just to check the relation between the generating functional and the diagrams, including keeping track of the coefficients.

2. For a scalar field theory, with $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$, show that the EOM are satisfied, up to a contact term:

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\langle T\phi(x)\phi(y)\rangle + \langle V'(\phi(x))\phi(y)\rangle = \alpha\delta(x-y).$$

   To do this problem, consider the functional integral and use the invariance of the functional integral under a change of variables. The change of variables is $\phi \to \phi + \epsilon f(x)$, where $f(x)$ is an arbitrary function of x, and ϵ is an infinitesimal parameter (drop terms of order ϵ^2 and higher). Derive in this way the above result (and determine the coefficient α). (Don't worry about the Jacobian in the $[d\phi]$ integration measure - it doesn't contribute.) The source term for ϕ is J, and you can think of $f(x)$ as a source for the EOM.

3. This exercise introduces the basic ingredient for defining functional integrals for fermionic fields (e.g. the electron).

   Define anticommuting (a.k.a. "Grassmann") numbers by the multiplication rule $\theta\eta = -\eta\theta$, so $\theta^2 = 0$. A function of a single real Grassmann variable has a simple Taylor's expansion, $f(\theta) = A + B\theta$, where A and B are constants. Similarly, for a function of two real Grassmann variables, we have $f(\theta, \eta) = A + B\theta + C\eta + D\theta\eta$. Grassmann integration over a real Grassmann variable is defined by $\int d\theta = 0$ and $\int d\theta\theta = 1$ (Grassmann integration acts the same as differentiation.) Rather than working with real grassmann variables, let's package two real grassmann variables into a single complex grassmann variable: $\theta = (\theta_1 + i\theta_2)/\sqrt{2}$. Then e.g. $\int d\theta^* d\theta \theta \theta^* = 1$.

Verify that $\int d\theta^* d\theta e^{-b\theta^* \theta} = b$, and more generally that

$$\prod_j \int d\theta_j^* d\theta_j e^{-(\theta^*, B\theta)} = \det B,$$

where $(\theta^*, B\theta) = \sum_{ij} B_{ij} \theta_i^* \theta_j$.