3/9/15 -3/13/15 week, Ken Intriligator's Phys 4D Lecture outline

• 4-vector dot product $a \cdot b = a_\mu b^\mu = a^\mu b^\nu \eta_{\mu\nu}$. If $a^\mu = (a^0, \vec{a})$, then $a_\mu = (a^0, -\vec{a})$. E.g. $x^\mu = (ct, \vec{x})$ and $x_\mu = (ct, -\vec{x})$. If a^μ has the usual Lorentz transformation, then a_μ has the inverse transformation ($\vec{v}_{rel} \to -\vec{v}_{rel}$), so $a_\mu b^\mu$ is invariant. Recall from HW 1 that $\partial^\mu = (\partial_{x^0}, -\nabla)$.

• Relativity and electromagentism. Can write the Lorentz force law as a relativisitc 4-vector equation $\frac{dp^\mu}{d\tau} = f^\mu = (f^0, \vec{f})$, with $f^0 = \gamma P$ and $\vec{f} = \gamma \vec{F}$. The Lorentz force law is $\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B})$, which we can rewrite as $\vec{f} = q(u^0 \vec{E} + \vec{u} \times \vec{B})$, where $u^\mu = (u^0, \vec{u}) = \frac{dx^\mu}{d\tau} = (\gamma, \gamma\vec{v})$. The equation for power gives $\frac{dp^0}{d\tau} = \frac{q}{c}\vec{u} \cdot \vec{E}$. These equations assemble into a 4-vector equation $\frac{dp^\mu}{d\tau} = f^\mu = \frac{q}{c} F^{\mu\nu} u_\nu$. Here $F^{\mu\nu} = -F^{\nu\mu}$, with $F^{0,i} = -E^i$ and $F^{12} = -B^3$, $F^{13} = B^2$, and $F^{23} = -B^1$.

Note that $f^\mu = m\frac{dv^\mu}{d\tau}$ and $u_\mu u^\mu = c^2$ implies that $u_\mu f^\mu = \frac{1}{2} m \frac{d(u_\mu u^\mu)}{d\tau} = 0$, and this is satisfied by the Lorentz force law above since $F^{\mu\nu} u_\mu u_\nu = 0$, thanks to $F^{\mu\nu} = -F^{\nu\mu}$.

• Under a Lorentz transformation, 4-vectors have $a^\mu = \sum \Lambda^\mu_{\nu'} a^{\nu'}$, and $F^{\mu\nu}$ behaves like that for each index, i.e. $F = \Lambda^T F' \Lambda$. Taking the two frames to have relative velocity v_{rel} along the $\hat{x}$ axis, this implies that (setting $c = 1$): $E_x = E'_x$, $B_x = B'_x$, and

$$\begin{pmatrix} E_y \\ B_z \end{pmatrix} = \begin{pmatrix} \gamma & \beta\gamma \\ \beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E'_y \\ B'_z \end{pmatrix}, \qquad \begin{pmatrix} E_z \\ B_y \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} \begin{pmatrix} E'_z \\ B'_y \end{pmatrix}.$$

(Note $\vec{B} \to \vec{E}$, $\vec{E} \to -\vec{B}$ symmetry). Examples illustrate this: the electric field of a line or plane of charge, vs the magnetic field if these charges are moving with velocity $\vec{v}$.

There are two invariant combinations $\vec{E}^2 - \vec{B}^2$ and $\vec{E} \cdot \vec{B}$. If $\vec{E} \cdot \vec{B} = 0$, can find a frame where either $\vec{E}' = 0$ or $\vec{B}' = 0$, depending on sign of $\vec{E}^2 - \vec{B}^2$.

• Maxwell's equations (in the Gaussian / CGS units that are nice for relativity)

$$\nabla \cdot E = 4\pi\rho \qquad \nabla \times B - \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c} J,$$

$$\nabla \cdot B = 0, \qquad \nabla \times E + \frac{1}{c}\frac{\partial B}{\partial t} = 0.$$

can be written as two 4-vector equations:

$$\frac{\partial}{\partial x^\mu} F^{\mu\nu} = \frac{4\pi}{c} j^\nu, \qquad \frac{\partial}{\partial x^\mu} \tilde{F}^{\mu\nu} = \frac{4\pi}{c} \tilde{j}^\nu = 0,$$

where $j^\mu = (c\rho, \vec{J})$ is the electric charge density and current density, which combine into a 4-vector, and $\tilde{F}^{\mu\nu}$ is related to $F^{\mu\nu}$ by $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$, and $\tilde{j}^\nu = 0$ reflects the non-observance of magnetic monopoles.

Charge conservation is invariant under Lorentz transformations;

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} \equiv \frac{\partial}{\partial x^\mu} j^\mu = 0$$

It is required by Maxwell's equations since $\partial_\mu \partial_\nu F^{\mu\nu} = 0$ by the symmetry of the two derivatives and antisymmetry of $F^{\mu\nu}$.

The $\frac{\partial}{\partial x^\mu} \tilde{F}^{\mu\nu} = 0$ Maxwell equations can be solved by introducing the scalar and vector potential, $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$, $\vec{B} = \nabla \times \vec{A}$, which can be written relativistically as $F^{\mu\nu} \equiv \partial^\mu A^\nu - \partial^\nu A^\mu$. Then Lorentz transformations of $\vec{E}$ and $\vec{B}$ then follow from ∂^μ and $A^\mu \equiv (\phi, \vec{A})$ being 4-vectors.

The wave equation for light as an electromagnetic wave in vacuum (so $j^\mu = 0$) can be written as $\partial_\mu \partial^\mu \vec{E} = \partial_\mu \partial^\mu \vec{B} = 0$, which is invariant since $\partial \cdot \partial = \partial' \cdot \partial'$ and the components of $\vec{E}$ and $\vec{B}$ mix according to the above Lorentz transformation, but will still satisfy the same wave equation in every inertial frame.

• Different example: uniformly accelerated motion along the x axis: find $a^\mu = \frac{du^\mu}{d\tau}$ such that $u \cdot u = c^2$, $u \cdot a = 0$, and $a \cdot a = -g^2$ for proper acceleration g. The solution has $v = dx/dt = c\tanh(g\tau/c)$, which reduces to $v \approx gt$ and $t \approx \tau$ for small velocities. Then $u^\mu = c(\cosh g\tau/c, \sinh g\tau/c, 0, 0)$ and $a^\mu = g(\sinh(g\tau/c), \cosh(g\tau/c), 0, 0)$; note that these satisfy all the conditions. Integrating, get $x = c^2 g^{-1}(\cosh(g\tau/c)$ and $t = cg^{-1}\sinh(g\tau/c)$. For $g\tau/c \ll 1$, get $t \approx \tau$ and $x \approx \frac{1}{2}gt^2$, i.e. the usual expressions for constant acceleration. The relativistic expressions asymptote for $g\tau/c \gg 1$ to $x \approx c^2 e^{g\tau/c}/2g$ and $t \approx ce^{g\tau/c}/2g$.

• Now, for a bit of general relativity. Emphasize $m_{inertial}$ and m_{grav} and the equivalence principle, leading to a_{grav} independent of mass. Led to Einstein's equivalence principle: no difference between gravity and gravity. Freely falling observer doesn't notice gravity. Accelerating observer sees something equivalent to gravity's "force." Einstein used this to replace gravity with spacetime curvature. What it means:

1. Replace $\eta_{\mu\nu}$ with a spacetime metric $g_{\mu\nu}$ (a.k.a. the "fabric of space-time"), which is affected by masses according to a differential equation called Einstein's equations (analog of Maxwells equations, but more complicated because it's not linear). Let's mention one solution of these equations: for a mass M at the origin, the solution is called the Schwarzschild metric (the gravity analog of the electric field of a point charge):

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = (1 - (2GM/r))(cdt)^2 - (1 - (2GM/r))^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

For $M = 0$, this reduces to the interval we've been discussing, $g_{\mu\nu} \to \eta_{\mu\nu}$, just written in spherical coordinates. For $M \neq 0$, this approximates to the interval we've been discussing when $r \gg 2GM$, where spacetime is approximately flat and unaffected by the mass M. For $2GM/r$ non-negligible, the effects of the curved spacetime have to be accounted for.

2. If there are no external forces, then $f^\mu = m\frac{Du^\mu}{d\tau} = 0$ says that 4-velocity is constant. Gravity is not added as an external force, here f^μ are just the forces other than gravity. Instead, gravity's effects are all in $g_{\mu\nu}$, and $g_{\mu\nu}$ is hiding in the $\frac{D}{d\tau}$ (which is why it's written here with D instead of d, as a reminder). This gives what is called the geodesic equation: the object moves along the extremal distance in space-time. Recall that the straight path in flat spacetime has longest proper length, and this is the path of a non-accelerated object. Likewise, the earth goes around the sun following the analog of a straight-line path, but in the curved spacetime metric $g_{\mu\nu}$ of the sun. We see an apparent inward acceleration, but the true relativistically defined acceleration, accounting for spacetime curvature, is actually zero.

• The $ds^2 = (1 - 2GM/r)(cdt)^2 + \ldots$ term means that clock rates depend on r. The clock records the wristwatch or proper time $ds^2 = (cd\tau)^2$, not the coordinate time dt. Suppose Alice is at r_A and Bob is at r_B, then $d\tau_{A,B} = \sqrt{1 - (2GM/r_{A,B})}dt$. Suppose Alice is at the top of the Eiffel tower and Bob is at the bottom, then $d\tau_A > d\tau_B$, Alice ages more. Atomic clocks are sufficiently precise to measure this difference, even for just 1 meter high difference in the earth's gravity field.

Suppose Alice drops photons down to Bob. Alice's photons have frequency $\omega_{emit} = 2\pi/\tau_A$, while the photons Bob receives have frequency $\omega_{receive} = 2\pi/\tau_B$, where the period is the corresponding proper time. Since $\tau_A > \tau_B$, $\omega_{receive} > \omega_{emit}$. This can be roughly understood as saying that photons at the bottom are gravitationally blueshifted because of they picked up energy in their fall, somewhat analogous to the gain in kinetic energy of a falling mass. Likewise, when a photon has to climb up, out of a gravitational potential, it'll be redshifted. Pound and Rebka directly measured this back in 1959 by sending photons down from the 4th floor of the Harvard Physics building to the basement (22 meters, about 74 feet). The key to measure the small frequency shift was to use the Mossbauer effect (found in 1958).

• Black holes: if $r_{object} < 2GM$, there is an event horizon at $r_H = 2GM$. According to the equivalence principle, an infalling observer can cross r_H without much drama, and the apparent singularity of ds^2 there is just a coordinate artifact of a global issue: the time and space coordinates become exchanged. Time now points into the actual singularity of the metric, which is at $r = 0$. There are many scientific debates about how to reconcile the equivalence principle with quantum mechanics, including the recent suggestion that there has to be a firewall at r_H to prevent some puzzles or potential mathematical contradictions regarding entropy and information.

- Cosmology:

$$ds^2 = (cdt)^2 - a(t)^2 d\vec{x}^2,$$

where the scale factor $a(t)$ is determined by Einstein's equations. Inflation or the big bang theory leads to a rapid growth of $a(t)$ in the early universe, from $a(t) \rightarrow \approx 0$ around 13.798 $\times 10^9$ years ago. Supernova observations show that $a(t)$ is accelerating in expansion, fitting with a cosmological constant (vacuum energy) $\Lambda \approx (10^{-3} eV)^4$. It's a great challenge for theorists to explain this tiny number: rough estimates are always much to big, by a factor of roughly 10^{120}. Although Λ is tiny, it accounts for over 68 percent of the Universe's energy budget, and that amount continues to grow as $a(t)$ increases, since more space means more energy of empty space.