2/4/13 Lecture outline

- Recall

$$\frac{\partial \mathcal{U}_{field}}{\partial t} + \frac{\partial}{\partial t}\mathcal{E}_{kin} + \nabla \cdot S = 0,$$

$$\frac{d}{dt}\left[\int_V dV(\mathcal{U}_{field} + \mathcal{E}_{kin})\right] + \int_{\partial V} \vec{S} \cdot d\vec{a} = 0.$$

$\vec{S} = c\vec{E} \times \vec{B}/4\pi$ is the energy flux density. Also $\int dV \vec{J} \cdot \vec{E}$ is the received mechanical power.

- Examples: solenoid with $\dot{I} \neq 0$. Work out $U = \frac{1}{2}LI^2$ and $\vec{E}\theta \sim \dot{I}$, so $\vec{S} \sim -I\dot{I}\hat{r}$. Verify energy conservation.

Now cylindrical resistor with conductivity σ, so $\vec{J} = \sigma\vec{E}$. Suppose I is constant, so $\vec{E}$ and $\vec{B}$ are constant. Work out $\vec{S} \sim -\hat{r}I^2/\sigma$. There is no change in field energy. Show $\int \vec{S} \cdot d\vec{a}$ agrees with power loss of resistor.

- For an electron, both q and $\vec{m} \neq 0$, so $\vec{S} = qc\vec{m} \times \vec{r}/4\pi r^6$. Note $\vec{S} \cdot \hat{r} = 0$.
- Last time: capacitor charging up, $U_{field} \approx \frac{1}{8\pi}\int dV \vec{E}^2 = Q^2/2C$, and $\int_{\partial V} \vec{S} \cdot d\vec{a} = -\frac{c}{4\pi}\Delta\phi \oint \vec{B} \cdot d\vec{\ell} = -\Delta\phi\dot{Q}$. Next example: starting up a solenoid.
- Field momentum density $\vec{\mathcal{P}}_{field} \equiv \vec{g} = \vec{S}/c^2 = \frac{1}{4\pi c}\vec{E} \times \vec{B}$, with $\vec{P}_{field} = \int dV \vec{g}$. We'll see in relativity this $\vec{g}/\vec{S}/c^2$ is related to $T^{\mu\nu}_{field} = T^{\nu\mu}_{field}$. It is also related to thinking about the field energy and momentum as carried by a stream of particles, photons: $\vec{S} = n_\gamma \vec{v}_\gamma E_\gamma$, where n_γ is the photon number density, and $\vec{v}_\gamma$ is their velocity and E_γ their energy. Likewise $\vec{\mathcal{P}}_{field} = n_\gamma \vec{p}_\gamma$. Recall that $\vec{p}_\gamma = \vec{v}_\gamma E_\gamma/c^2$.

Using the force law for particles q_i, get

$$\frac{d}{dt}\vec{P}_{mech} = \int dV(\rho\vec{E} + \frac{1}{c}\vec{J} \times \vec{B}) \equiv \int dV \frac{\partial}{\partial t}\vec{\mathcal{P}}_{mech}.$$

Momentum conservation:

$$\frac{d}{dt}\left[\int dV(\vec{\mathcal{P}}_{field} + \vec{\mathcal{P}}_{mech})_i\right] + \oint_{\partial V} T_{ij}da^j = 0.$$

Find it from using Maxwell's equations to show

$$\frac{\partial}{\partial t}g_i = -(\rho\vec{E}_i + \frac{1}{c}(\vec{J} \times \vec{B})_i) - \frac{\partial}{\partial x_j}T_{ij}$$

with

$$T_{ij} = \frac{1}{4\pi}\left(\tfrac{1}{2}(E^2 + B^2)\delta_{ij} - E_iE_j - B_iB_j\right)$$

the flux of field momentum. The sign is such that e.g. T_{ii} on a surface is the inward pressure.

Examples: conductor and attractive force. Current sheets (opposite sign): repulsive force. Solenoid along the $\hat{z}$ axis: $T_{xx} = T_{yy} = B^2/8\pi$, and $T_{zz} = -B^2/8\pi$, and pressure on inner surface of the solenoid is $\vec{F}_i/A = -T_{ij}\hat{n}^i_{out}$, with $\hat{n}_{out} = -\hat{r}$ the outward normal to the inner surface of the solenoid. Upshot: an outward pressure of $B^2/8\pi$.

• Field angular momentum density $\vec{\mathcal{L}}_{field} = \vec{r} \times \vec{\mathcal{P}}_{field}$. Particle's angular momentum density $\vec{\mathcal{L}}_{matter} = \vec{r} \times \vec{\mathcal{P}}_{mech}$. Show

$$\frac{\partial}{\partial t}\left(\mathcal{L}_{field,i} + \mathcal{L}_{matter,i}\right) = -\frac{\partial}{\partial x_m} M_{im},$$

with $M_{im} = \epsilon_{ijk} x_j T^{field}_{km}$ the field angular momentum flux. So

$$\frac{d}{dt}\left[\int dV(\vec{\mathcal{L}}_{field} + \vec{\mathcal{L}}_{matter})_i\right] + \oint_{\partial V} M_{ij} da_j = 0$$

E.g. magnetic monopole at origin and electric charge at displaced location: $\vec{B} = g\hat{r}/r^2$, $\vec{E} = q(\vec{r} - \vec{a})/|\vec{r} - \vec{a}|^3$. Compute $\vec{L}_{field,z} = eg/c = n\hbar/2$, gives Dirac quantization.

• Example: setup from last lecture, where $L = \frac{1}{2}I\dot{\theta}^2 + \frac{Q}{c}R\dot{\theta}A_\theta$, so $p_\theta = I\dot{\theta} + \frac{QR}{c}A_\theta$ is constant. The first term is the mechanical angular momentum. Can work out that the second term gives the field angular momentum, $\vec{L}_{field} = \hat{z}\frac{Q}{c}A_\theta R$.