1/16/13 Lecture outline

$\star$ Garg chapter 4.

• Last time: we can always compute $\vec{B}$ from the curl of $\vec{A}$. But in regions with $\vec{J}=0$, we can also compute it from $\vec{B}=-\nabla\phi_{mag}$. We saw that for a point magnetic dipole: $\phi_{mag}=\vec{m}\cdot\vec{r}/r^3$ or $\vec{A}=\vec{m}\times\vec{r}/r^3$. Note that they give the same $\vec{B}$ away from the origin, but differ by delta function contributions at $\vec{r}=0$ (where the current is hiding for a point dipole). Follows from $\nabla_i\nabla_j r^{-1}=(3x_ix_j-r^2\delta_{ij})/r^5-\frac{4\pi}{3}\delta_{ij}\delta(\vec{r})$. Writing $\vec{B}^{wrong}=-\nabla\phi_{mag}$ has a $-(4\pi/3)$ coefficient of $\delta(\vec{r})$ whereas $\vec{B}^{right}=\nabla\times\vec{B}$ has a $+(8\pi)/3$ coefficient (which is the correct one).

(Aside: Q: How does ϕ_{mag} or $U=-\vec{m}\cdot\vec{B}_{ext}$ fit with $\vec{F}_{mag}=\frac{q}{c}\vec{v}\times\vec{B}$ doing no work? A: We'll see it better next week, when we discuss EMF.)

• Last time: observe $U=-\vec{m}\cdot\vec{B}$ and $\tau=\vec{m}\times\vec{B}$. Want to compare this with the torque on a current loop. For a small square loop, it is straightforward to obtain $\tau_{loop}=\vec{m}_{loop}\times\vec{B}$, with $\vec{m}_{loop}=I\vec{a}_{loop}/c$. Let's now do it for a general loop: $\tau=\int\vec{r}\times(Id\vec{\ell}\times\vec{B})/c$. For a general current density, we have $\vec{\tau}=\int dV\vec{r}\times(\vec{J}\times\vec{B})/c$. Now show $\tau=\vec{m}\times\vec{B}$, with $\vec{m}=\frac{1}{2c}\int dV\vec{r}\times\vec{J}=\frac{I}{2c}\int\vec{r}\times d\vec{\ell}=\frac{I}{c}\vec{a}$. Need to explain the cross product rearrangement and the factor of $\frac{1}{2}$. Illustrates some occasionally useful identities. Using the above, have

$$c\vec{\tau}=\int dV\left(\vec{J}(\vec{r}\cdot\vec{B})-\vec{B}(\vec{r}\cdot\vec{J})\right).$$

We want to show this equals

$$\tfrac{1}{2}\int dV(\vec{r}\times\vec{J})\times\vec{B}=\tfrac{1}{2}\int dV\left(\vec{J}(\vec{r}\cdot\vec{B})-\vec{r}(\vec{J}\cdot\vec{B})\right).$$

Use identities that follow from $\nabla\cdot\vec{J}=0$ (to be compatible with charge conservation $\nabla\cdot\vec{J}+\frac{\partial\rho}{\partial t}=0$ in the static case; fits with point charges moving at constant velocities). Then, for localized currents and any functions $f(\vec{r})$ and $g(\vec{r})$,

$$\int dV(f\vec{J}\cdot\nabla g+g\vec{J}\cdot\nabla f)=0$$

(via integrating 2nd term by parts). Setting $f=x_i$ and $g=x_j$, get $\int dVx_{(i}J_{j)}$, where the parenthesis means symmetrize. Use this to show the above (eg. for constant $\vec{B}$).

• Recap: we use $\vec{F}=q\vec{v}\times\vec{B}/c$ to connect current loops to magnetic dipoles, with $\vec{m}_{loop}=I\vec{a}/c=\frac{I}{2c}\int\vec{r}\times d\vec{l}=\frac{1}{2c}\int dV\vec{r}\times\vec{J}$. Now we recall the magnetic field due to $\vec{m}$

and use this relation to find $\vec{B}$ from I or $\vec{J}$, e.g. $\vec{B}(\vec{r}) = \nabla \times \int I d\vec{a}' \times (\vec{r} - \vec{r}')/c|\vec{r} - \vec{r}'|^3$. Massage to obtain the formula of Biot and Savart

$$\vec{B}(\vec{r}) = \frac{I}{c} \oint \frac{d\vec{\ell}' \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{1}{c} \int d^3\vec{x}' \frac{\vec{j}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}|^3}$$

We can now show that this has $\nabla \cdot \vec{B} = 0$ and $\oint \vec{B} \cdot d\vec{\ell} = 4\pi I_{encl}/c$.

• The above is a roundabout, "scenic detour" way to Maxwell's equation for $\vec{B}$ in the magnetostatic case $(\partial \vec{E}/\partial t = 0)$, $\nabla \times \vec{B} = 4\pi \vec{J}/c$, where $\vec{J} = \sum_n q_n \vec{v}_n \delta(\vec{r} - \vec{r}_n)$. Use $\vec{B} = \nabla \times \vec{J}$ and $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla . \vec{A}) - \nabla^2 \vec{A}$, and take $\nabla \cdot \vec{A} = 0$ (using the gauge freedom) to see that we want to solve $-\nabla^2 \vec{A} = 4\pi \vec{J}/c$. We can solve this using the same Green's function for the Laplacian that we used last time:

$$\vec{A}(\vec{x}) = \frac{1}{c} \int d^3\vec{x}' \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} = \sum_n q_n \frac{\vec{v}_n}{c|\vec{x} - \vec{x}_n|}.$$

Taking the curl then gives the above BS formula.

• Can do a multiple expansion for above expression for $\vec{A}$. The monopole term vanishes, since $\vec{J}$ is has no divergence. The leading term is the dipole term, which gives

$$\vec{A}(\vec{r}) \approx \frac{\vec{m} \times \vec{r}}{r^3}, \qquad \vec{m} = \frac{1}{2c} \int \vec{r} \times \vec{j}(\vec{r}) d^3x.$$

Writing $\vec{j} = \sum_i q_i \vec{v}_i \delta(\vec{r} - \vec{x}_i)$, this gives $\vec{m} = \sum_i q_i \vec{r}_i \times \vec{v}_i / 2c = q\vec{L}/2Mc$.

• Example: loop of current of radius a in the $\cos\theta = 0$ plane. Can evaluate the integral for $\vec{A}$ in terms of elliptic integrals or the spherical harmonics (Jackson). Find e.g. $B_r = (2\pi I a/rc) \sum_{n=0}^{\infty} (-1)^n ((2n+1)!!/2^n n!) r_<^{2n+1} r_>^{-(2n+2)} P_{2n+1}(\cos\theta)$, where $r_<$ is the smaller of a or r, and a similar expression for B_θ. For $r \gg a$, keep just the $n = 0$ term, giving a dipole field with $m = \pi I a^2/c$.

• Find $\vec{B}$ in symmetric examples, using $\oint B \cdot d\vec{\ell} = 4\pi I_{enc}/c$, e.g. infinite straight wire, $\vec{B} = 2I\hat{e}_\phi / r_\perp$. Solenoid. B_ϕ for torus solenoid.