2/27/13 Lecture outline

• Today's topic: motion of charged particles in (external) $\vec{E}$ and $\vec{B}$ fields.

• Lagrangian: $L = L_0 + \frac{q}{c}\vec{v}\cdot\vec{A} - q\phi$, where L_0 is the free-particle Lagrangian. For the non-relativistic case, $L_0 \approx \frac{1}{2}m\vec{v}^2$. In relativity, $L_0 = -mc^2\sqrt{1-v^2/c^2}$. We'll understand it better next week, but can still use it now. So note that

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \vec{p}_0 + \frac{q}{c}\vec{A}, \qquad H = \vec{p}\cdot\vec{v} - L = \gamma mc^2 + q\phi,$$

with $\vec{p}_0 = \partial L_0/\partial\vec{v} = \gamma m\vec{v}$. The EL equations of motion are

$$\frac{d}{dt}\vec{p} = \frac{d}{dt}(\vec{p}_0 + \frac{q}{c}\vec{A}) = \frac{\partial L}{\partial \vec{r}} = \frac{q}{c}\nabla(\vec{v}\cdot\vec{A}) - q\nabla\phi$$

Since $\vec{A} = \vec{A}(t, \vec{r}(t))$, $\frac{d\vec{A}}{dt} = \frac{\partial A}{\partial t} + (\vec{v}\cdot\nabla)\vec{A}$. Recalling $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$ and $\vec{B} = \nabla\times\vec{A}$, the above can be rewritten as the Lorentz force law:

$$\frac{d\vec{p}_0}{dt} = qE + \frac{q}{c}\vec{v}\times\vec{B}.$$

• Relativistic example: charged particle in a static uniform $\vec{E}$, with $\vec{B} = 0$. Take $\vec{E} = E\hat{z}$. Then

$$L = -mc^2\sqrt{1 - c^{-2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)} + qE_0 z.$$

Note $p_i = \partial L/\partial v_i = \gamma m\dot{x}_i$. Since x and y and t don't enter, we have three conserved quantities:

$$p_x = \gamma m\dot{x}, \qquad p_y = \gamma m\dot{y}, \qquad \mathcal{E} = \gamma mc^2 - qEz.$$

And $\dot{p}_z = qE_0$. So $p_z = qEt$. Suppose that at time $t = 0$, $\vec{p} = p_0\hat{x}$ and $\vec{x} = 0$. Then the conserved energy is $\mathcal{E} = \sqrt{c^2p_0^2 + (mc^2)^2}$ and conservation of p_x and p_y give $p_0 = \gamma m\dot{x}$ and $y = \dot{y} = 0$. Use a change of variables, $\gamma \equiv \frac{dt}{d\tau}$, so $p_0 = m\frac{dx}{d\tau}$, which integrates to $x = p_0\tau/m$. Also $p_z = \gamma m\dot{z} = qEt$ now gives $m\frac{dz}{d\tau} = qEt$. Taking $\frac{d}{d\tau}$ of the $\mathcal{E}$ conservation equation gives $\frac{d^2t}{d\tau^2} = (qE/mc^2)\frac{dz}{d\tau} = (qE/mc)^2 t$. Integrating, we get $t = t_0\sinh(qE\tau/mc)$, and $\gamma = \frac{dt}{d\tau} = (qEt_0/mc)\cosh(qE\tau/mc)$. We can determine t_0 since at $\tau = t = 0$, $\gamma = \mathcal{E}/mc^2$, so $t_0 = \mathcal{E}/qEc$. Using $\tau/m = x/p_0$, we get

$$t = \frac{\mathcal{E}}{qEc}\sinh(qEx/p_0c), \qquad z = \frac{\mathcal{E}}{qE}(\cosh(qEx/p_0c) - 1).$$

For τ and x small, we have $x \approx v_0 t$, with $v_0 = p_0c^2/\mathcal{E}$ and $z \approx \frac{1}{2}at^2$, with $a = qE/m$. For late times,

$$t \approx \frac{\mathcal{E}}{2qEc}e^{qEx/p_0c}, \qquad z \approx \frac{\mathcal{E}}{2qE}e^{qEx/p_0c} \to \frac{dz}{dt} \approx c$$

• Motion in a static, uniform $\vec{B}$. Consider the nonrelativistic case. $\dot{\vec{v}} = -\frac{q}{mc}\vec{B}\times\vec{v}$, so $\vec{\Omega}_c = -q\vec{B}/mc$. In the relativistic case, use $\dot{\vec{p}} = (\mathcal{E}\dot{\vec{v}})/c^2$, with $\mathcal{E}$ a constant since $\vec{B}$ does no work. So $\dot{\vec{v}} = -(qc\langle B\rangle/\mathcal{E})\times\vec{v}$, so $\vec{\Omega}_c = -qc\vec{B}/\mathcal{E}$.

We can write $\vec{B} = B\widehat{z}$ in terms of $\vec{A} = Bx\widehat{y}$, so

$$L = -mc^2\sqrt{1-v^2/c^2} + \frac{qB}{c}v_y Bx.$$

The constants of the motion are

$$p_y = \gamma m v_y + \frac{qBx}{c}, \qquad p_z = \gamma m v_z, \qquad \mathcal{E} = \gamma mc^2.$$

and the equation of motion is $\frac{d}{dt}(\gamma m v_x) = qBv_y/c$. See rotation in x, y plane with $\Omega = qB/\gamma mc$.

• Motion when $\vec{E}$ and $\vec{B}$ are both constant and perpendicular. Take $\phi = xV/d$ and $\vec{A} = \widehat{y}Bx$. Suppose an electron is released from at rest at $x = 0$, find minimum V such that it can reach $x = d$. Use

$$L = -mc^2\sqrt{1-v^2/c^2} - \frac{eB_0}{c}xv_y + e\frac{x}{d}V.$$

The constants are

$$p_y = \gamma m v_y - \frac{e}{c}B_0 x = 0, \qquad p_z = \gamma m v_z = 0, \qquad \gamma mc^2 - e\frac{x}{d}V = mc^2$$

Can use these to find the minimum V.

• Take $\vec{E} = E_0\widehat{y}$ and $\vec{B} = B_0\widehat{z}$, and suppose $E_0 \ll B_0$ and treat it nonrelativistically. There is a drift velocity, with in general $\langle\vec{v}\rangle = c\vec{E}\times\vec{B}/B^2$. Looks bizarre: drifts in direction $\perp$ to the fields. We'll see later how it makes sense in terms of boosted frame fields. Show directly: solving the equations gives

$$v_y = \frac{eE_0}{m\Omega_c}\sin\Omega_c t, \qquad v_x = -\frac{eE_0}{m\Omega_c}(\cos\Omega_c t - 1),$$

with the last term the drift velocity.

• Motion in a slowly varying $\vec{B}(t) = B(t)\widehat{z}$. Compute $\Delta(\frac{1}{2}mv_\perp^2) \approx -e\oint d\vec{\ell}\cdot E = \frac{1}{c}\int\frac{\partial\vec{B}}{dt}\cdot d\vec{a} \approx \frac{e}{c}\dot{B}\pi v_\perp^2/\Omega_c^2$. Now $(2\pi/\Omega_c)\dot{B} = \Delta B$, so we get $\Delta(mv_\perp^2/2B) \approx 0$. Adiabatic invariant.

• Drift's in weakly inhomogeneous $\vec{B}$ (Alfven). Particles have cyclotron spiral rotations around $\vec{B}$ field lines, averaging over them leads to drift in direction $\perp$ to $\vec{B}$ (see O'Neil's notes). E.g. torus solenoid, this drift leads to collection of + charges on top and − charges on bottom, with $\vec{B}$ into page. Results in $\vec{E}$ such that $\vec{E}\times\vec{B}$ drift pushes plasma out in the radial direction. Plasma tokamak or mirror bottles. Earth's Van Allen belts