2/25/13 Lecture outline

• Last time: In Lorentz gauge, Maxwell's equations are solved via $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$, $\vec{B} = \nabla\times\vec{A}$, with

$$\partial^2\phi(\vec{r},t) = 4\pi\rho(\vec{r},t), \qquad \partial^2\vec{A}(\vec{r},t) = \frac{4\pi}{c}\vec{J}(\vec{r},t). \tag{1}$$

We found the Greens function and this led to solutions

$$\phi(\vec{r},t) = \int d^3\vec{r}'\frac{\rho(\vec{r}',t-R/c)}{R}, \qquad \vec{A}(\vec{r},t) = \frac{1}{c}\int d^3\vec{r}'\frac{\vec{J}(\vec{r}',t-R/c)}{R}.$$

For the case of a uniformly moving charge ($\vec{a}=0$), we found this led to

$$\phi(\vec{r},t) = \gamma\frac{q}{[(\gamma x-\gamma vt)^2+y^2+z^2]^{1/2}}, \qquad \vec{A} = \frac{\vec{v}}{c}\phi \tag{2}$$

Then found $\vec{B} = \frac{1}{c}\vec{v}\times\vec{E}$ and

$$\vec{E}(\vec{r},t) = q\gamma^{-2}\frac{\vec{r}-\vec{r}'(t)}{((x-vt)^2+\gamma^{-2}(y^2+z^2))^{3/2}}.$$

• Now consider q with general motion $\vec{r}_0(t)$. So

$$\rho(\vec{r},t) = q\delta(\vec{r}-\vec{r}_0(t)), \qquad \vec{J}(\vec{r},t) = q\dot{\vec{r}}_0(t)\delta(\vec{r}-\vec{r}_0(t).$$

$$\phi(\vec{r},t) = q\int d^3\vec{r}'dt'\frac{1}{|\vec{r}-\vec{r}'|}\delta(\vec{r}'-\vec{r}_0(t'))\delta(t'-t+\frac{1}{c}|\vec{r}-\vec{r}_0(t')|).$$

Now $\delta(t'-\ldots) = \delta(t'-t_r)\zeta$, where

$$\zeta = 1-\frac{1}{c}\frac{dR_a}{dt} = (1+\frac{1}{c}\frac{dR_a}{dt_r})^{-1} = \frac{1}{1-\vec{\beta}_r\cdot\widehat{R}_a}$$

where $\vec{R}_a(t) = \vec{r}-\vec{r}_a(t)$, and $\vec{r}_a(t) = \vec{r}_0(t_r)$, and $\vec{v}_r = \frac{d}{dt}\vec{r}_0(t)|_{t=t_r}$ and $\vec{\beta}_r = \vec{v}_r/c$. Note $\frac{d}{dt} = \zeta\frac{d}{dt_r}$ so e.g. $\vec{v}_a = \zeta\vec{v}_r$, where $\vec{v}_a = -\frac{d}{dt}\vec{R}_a(t)$. So get the Lienard-Wiechert potentials

$$\phi(\vec{r},t) = \frac{q\zeta}{R_a} = \frac{q}{R_a-\beta_r\cdot\vec{R}_a}, \qquad \vec{A}(\vec{r},t) = \frac{q\vec{\beta}_r}{R_a-\vec{\beta}_r\cdot\vec{R}_a}.$$

Plug these in to get $\vec{E}$ and $\vec{B}$. Find $\vec{B} = \widehat{R}_a\times\vec{E}$ and (Heaviside-Feynman)

$$\vec{E} = q\frac{\widehat{R}_a}{R_a^2} + q\frac{R_a}{c}\frac{d}{dt}\frac{\widehat{R}_a}{R_a^2} + q\frac{1}{c^2}\frac{d^2}{dt^2}\widehat{R}_a.$$

Alternatively, can write it as

$$\vec{E} = \frac{q(1-\beta_r^2)}{(R_a - \vec{\beta}_r \cdot R_a)^3}(\vec{R}_a - \vec{\beta}_r R_a) + \frac{q}{c^2(R_a - \vec{\beta}_r \cdot \vec{R}_a)^3}(\dot{\vec{v}}_r \times (\vec{R}_a - \vec{\beta}_r R_a)) \times \vec{R}_a.$$

• Consider charges moving in some localized region, with the observer far away. For static charges, the leading contribution to $\vec{E}$ (monopole term) is $\sim 1/R^2$. For moving charges, the leading term is $1/R$: $\vec{E} \approx -\frac{q}{Rc^2}\frac{d^2}{dt^2}\vec{r}_{0,\perp}(t - R/c)$. For a collection of charges, this gives more generally at leading order in large R:

$$\vec{E} \approx \frac{1}{Rc^2}\widehat{r} \times (\widehat{r} \times \frac{d^2}{dt^2}\vec{d}_{ret}), \qquad \vec{B} \approx \widehat{r} \times \vec{E}.$$

Let's derive it quickly again, in a way that you can easily remember for the leading contribution, far from the source

$$\vec{A}(\vec{r},t) \approx \frac{1}{c}\int d^3\vec{r}' \frac{1}{|\vec{r} - \vec{r}'|}\vec{J}(\vec{r}', t - |\vec{r} - \vec{r}'|/c), \qquad \vec{J} \approx \sum_i q_i \dot{\vec{r}}_i(t - r/c)\delta^3(\vec{r}' - \vec{r}_i),$$

Plug into $\vec{A}$, and Taylor expand $1/|\vec{r} - \vec{r}'|$ for large r, get

$$\vec{A} \approx \frac{1}{rc}\dot{\vec{d}}(t - r/c) + \frac{1}{2rc^2}\frac{d^2}{dt^2}Q_{ij}\widehat{r}_j \widehat{e}^i + \frac{1}{rc}(\dot{\vec{m}} \times \widehat{r}),$$

where recall $\vec{m} = \frac{1}{2c}\int d^3\vec{r}'\vec{r}' \times \vec{J}$. Gives

$$\vec{E} \approx \frac{1}{rc^2}\left[\widehat{r} \times (\widehat{r} \times \frac{d^2}{dt^2}\vec{d}_{ret}) + (\vec{d} \to \vec{m}) + (\vec{d} \to \frac{1}{2c}\frac{d\vec{Q}}{dt})\right].$$

Let's keep just the dipole term.

Then $\vec{S} = \frac{c}{4\pi}E^2\widehat{n}$ falls of as $1/R^2$, and

$$\frac{dP}{d\Omega} = \frac{1}{4\pi c^3}|(\widehat{r} \times \frac{d^2\vec{d}_{ret}}{dt^2}|^2$$

Integrating over solid angle, $P = \frac{2}{3c^2}\frac{d^2\vec{d}_{ret}}{dt^2}^2$

• Fourier transform $t \to \omega$, the above general expressions become

$$\phi_\omega(\vec{r}) = \int d^3x' \frac{e^{i\omega R/c}}{R}\rho_\omega(\vec{r}'), \qquad \vec{A}_\omega(\vec{r}) = \int d^3x' \frac{e^{i\omega R/c}}{R}\vec{j}_\omega(\vec{r}')/c.$$

Then $\vec{E}_\omega = -\nabla\phi_\omega + i\omega\vec{A}_\omega/c$ and $\vec{B}_\omega = \nabla \times \vec{A}_\omega$.

• Far zone: $|\vec{R}-\vec{r}'| \approx R - \vec{r}'\widehat{R}$ and $t_{ret} \approx t - (R - \vec{r}'\widehat{R})/c$ Get

$$\phi_\omega(\vec{R}) \approx \frac{e^{ikR}}{R}\rho_{\vec{k},\omega}, \qquad \vec{A}_\omega(\vec{R}) \approx \frac{e^{ikR}}{R}\vec{j}_{\vec{k},\omega}/c,$$

where $\vec{k} = \omega\widehat{R}/c$.

Dipole approximation: $\vec{j}_{\vec{k},\omega} \approx (\dot{\vec{d}})_\omega = -i\omega\vec{d}_\omega$. Then get $\vec{E}_\omega \approx k^2 \vec{d}_{\omega,\perp} e^{ikR/R}$, which in t gives what we mentioned above, $\vec{E} \approx -\frac{1}{Rc^2}\frac{d^2}{dt^2}\vec{d}_{ret,\perp}$.

Get higher multipoles by expanding $e^{-i\vec{k}\cdot\vec{r}}$ in the FT from $\vec{j}_\omega(\vec{r})$ to $\vec{j}_{\vec{k}\omega}$. Writing $e^{-i\vec{k}\cdot\vec{r}} \approx 1 - i\vec{k}\cdot\vec{r} + \ldots$, the 1 term gives the dipole approximation. The next term give contributions to $\vec{E}$ and $\vec{B}$ involving d^3/dt^3 of the quadrupole tensor. Etc.

• Antennae. Center fed example, $I = I_0 \cos\omega t \to \frac{d}{dt}\vec{d} = I_0 a \cos\omega t\widehat{z}$. Far field: $|\vec{E}| \approx \frac{I_0 a\omega}{Rc^2}\sin\omega t_r \sin\theta$. Better: $\vec{j} = I_0 \sin(\frac{1}{2}ka - k|z|)\cos\omega t\delta(x)\delta(y)\widehat{z}$. Leads to $\frac{d}{dt}\vec{d} \approx \frac{1}{4}I_0 ka^2 \cos\omega t\widehat{z}$.

Another example: dipole rotating in plane $z=0$, so $\frac{d^2\vec{d}}{dt^2} = -\omega^2\vec{d}$. Work out $\frac{d^2\vec{d}}{dt^2} \times \widehat{r}$.

Also, angular momentum radiated: $\frac{dL_i}{dt} = \oint M_{ij}\widehat{r}_j d^2s$, with $M_{ij} = \epsilon_{ikm}T_{jk}x^m$. Get e.g. for rotating dipole in plane $z=0$, $\frac{d\vec{L}}{dt} \approx \frac{2}{3c^3}d^2\omega^3\widehat{z}$, and $\dot{L}/\dot{\mathcal{E}} = 1/\omega$, suggestive of photons.

• Radiation reaction: an accelerating charge radiates power $P \approx \frac{2}{3}\frac{e^2}{c^3}\vec{a}^2$. This leads to a backreaction additional force needed to accelerate the charge.