2/20/13 Lecture outline

• Last time: considered solving $\partial^2\psi = 0$ in vacuum, using Greens functions for getting the boundary conditions on some surface. Today we'll solve the equations with source terms via Greens functions.

• In Lorentz gauge, Maxwell's equations are solved via $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$, $\vec{B} = \nabla\times\vec{A}$, with

$$\partial^2\phi(\vec{r},t) = 4\pi\rho(\vec{r},t), \qquad \partial^2\vec{A}(\vec{r},t) = \frac{4\pi}{c}\vec{J}(\vec{r},t). \tag{1}$$

We'd like to find a Green's function in space and time:

$$\partial^2 G(\vec{r},t;\vec{r}',t') = 4\pi\delta(\vec{r}-\vec{r}')\delta(t-t').$$

Translation symmetry: $G = G(\vec{R},\tau)$, where $\vec{R} \equiv \vec{r}-\vec{r}'$ and $\tau = t-t'$. Fourier transform in τ:

$$G(\vec{R},\tau) = \int \frac{d\omega}{2\pi} e^{-i\omega\tau}\tilde{G}(R,\omega)$$

then solutions are

$$\tilde{G}^{\pm} = \frac{e^{\pm ikR}}{R}, \qquad \rightarrow \qquad G^{\pm}(\vec{R},\tau) = \frac{1}{R}\delta(\tau \mp R/c) \equiv G_{ret,adv}.$$

The ret case connects a field at some time to the behavior of the source in the past, which is what we want. The adv case connects to the behavior of the source in the future, which we typically don't want[1]. So we get the Lorentz gauge soln's, which give the fields in terms of the charges' location at the earlier time

$$\phi(\vec{r},t) = \int d^3\vec{r}'\,\frac{\rho(\vec{r}',t-R/c)}{R}, \qquad \vec{A}(\vec{r},t) = \frac{1}{c}\int d^3\vec{r}'\,\frac{\vec{J}(\vec{r}',t-R/c)}{R}.$$

• Example: uniformly moving charge $\rho = q\delta(\vec{r}-vt\hat{x})$, $\vec{J} = qv\hat{x}\delta(\vec{r}-vt\hat{x})$.

$$\phi(\vec{r},t) = q\int d^3\vec{r}'\,\frac{1}{|\vec{r}-\vec{r}'|}\delta(\vec{r}'-vt\hat{x}+\frac{v}{c}|\vec{r}-\vec{r}'|\hat{x}) = \frac{q}{(R-\frac{v}{c}(x-x'))_{ret}}.$$

We used $\delta(x'-vt+\frac{v}{c}R) = \delta(x'-x'_{ret})/(1+\frac{v}{c}\frac{dR}{dx'})$ and x'_{ret} solves $x'_{ret} = vt - \frac{v}{c}R_{ret}$, with $R_{ret} = \sqrt{(x-x'_{ret})^2+y^2+z^2}$. So x'_{ret} satisfies a quadratic equation, which leads to

$$x'_{ret} = -\gamma^2\frac{v^2}{c^2}x + \gamma^2 vt - \frac{v}{c}\gamma^2\sqrt{(x-vt)^2+\gamma^{-2}(y^2+z^2)},$$

[1] Aside: in quantum field theory, the Feynman propagator is a superposition of adv and ret cases: "anti-matter travels backwards in time".

with $\gamma \equiv 1/\sqrt{1-v^2/c^2}$. The denominator above is simplified as

$$R_{ret}-\frac{v}{c}(x-x')_{ret} = -\frac{v}{c}(x-vt)+R_{ret}\gamma^{-2} = ct-\frac{v}{c}x-\frac{c}{v}x'_{ret}\gamma^{-2} = \sqrt{(x-vt)^2+\gamma^{-2}(y^2+z^2)}.$$

After a lot of work, we derive the simple answer:

$$\phi(\vec{r},t) = \gamma\frac{q}{[(\gamma x-\gamma vt)^2+y^2+z^2]^{1/2}} \tag{2}$$

Alternatively, as in the book, we can get the answer by Fourier transform:

$$\phi(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3}\phi_{\vec{k}}e^{i\vec{k}\cdot\vec{r}}e^{-k_x vt}$$

since the solution depends only on $x-vt$, and we plug in the wave equation with source to get

$$(k^2-\frac{v^2}{c^2}k_x^2)\phi_{\vec{k}} = 4\pi q$$

solving for $\phi_{\vec{k}}$ and doing the Fourier integrals again leads to (2). Likewise, $\vec{J}$ yields

$$\vec{A}(\vec{r},t) = \frac{q\vec{v}}{c(R-\frac{\vec{v}\cdot\vec{R}}{c})_{ret}} = \gamma\vec{v}\frac{q}{[(\gamma x-\gamma vt)^2+y^2+z^2]^{1/2}} \tag{3}$$

Whenever a lot of work leads to a simple answer, we should look for a simpler explanation. We'll later understand (2) and (3) as immediately coming from the Lorentz boost of the potentials from the particle's rest frame.

- Compute now $\vec{E}$ and $\vec{B}$. Note $\vec{E} = -\nabla\phi+\frac{1}{c^2}\vec{v}(\vec{v}\cdot\nabla\phi)$ and $\vec{B} = \frac{1}{c}\vec{v}\times\vec{E}$. Get

$$\vec{E}(\vec{r},t) = q\gamma^{-2}\frac{\vec{r}-\vec{r}'(t)}{((x-vt)^2+\gamma^{-2}(y^2+z^2))^{3/2}}.$$

Note radial but not spherically symmetric. Let $\vec{R} = \vec{r}-\vec{r}'(t)$. When $\vec{R}\perp\vec{v}$, get $\vec{E} = \gamma q\vec{R}/R^3$. When $\vec{R}||\vec{v}$ get $\vec{E} = \gamma^{-2}q\vec{R}/R^3$. Pancake effect.