2/13/13 Lecture outline

• Continuing from last time, today's topic is solving the D'Alembertian eq, which we saw last time governs $\vec{E}$ and $\vec{B}$ without sources. With sources, we'll see the D'Alembertian eq again, this time for the potentials ϕ and $\vec{A}$, with sources on the RHS from the ρ and $\vec{J}$. Let's first say some general things about the case without sources, $\partial^2\psi(\vec{r},t)=0$. This is a wave equation, giving propagation of waves at $v=c$! We saw that last time with electromagnetic plane waves in vacuum: $\omega=ck$, which says that the wave moves with $v_\phi=\omega/k=c=d\omega/dk=v_g$. Any function like $\psi=f(x-ct)$ satisfies $\partial^2\psi=0$. Today we'll consider more general kinds of solutions.

Last time: consider the wave equation $\partial^2\psi(\vec{r},t)$, which the components of $\vec{E}$ and $\vec{B}$ satisfy. Taking $\psi=Ref(\vec{r})e^{-ickt}$, get $(\nabla^2+k^2)f(\vec{r})=0$. An example is the plane wave, $f=e^{-\vec{k}\cdot\vec{r}}$. Find that e^{ikr}/r is a spherical wave solution, with source term at the origin:

$$(\nabla^2+k^2)\frac{e^{ikr}}{r}=\frac{1}{r^2}\frac{d}{dr}(r^2\frac{d}{dr}\frac{e^{ikr}}{r})=-4\pi\delta(\vec{r}).$$

• Other examples of solutions of the wave equation, e.g. spherical waves. Write complex $\vec{E}$ and $\vec{B}$, with the understanding to take the real part at the end. Consider wave with definite wavelength (monochromatic): $\vec{E}=\vec{E}_0(r)e^{ik\psi(r)-i\omega t}$, $\vec{B}=\vec{B}_0(r)e^{ik\psi(r)-i\omega t}$; let's not bother to keep writing the $e^{-i\omega t}$, just remember it's there. $\psi(r)$ is the eikonal. We're going to do an expansion for small wavelength, so $r\gg\lambda$. E.g. $\nabla\cdot\vec{B}\approx ikB_0\cdot\nabla\psi e^{ik\psi}$, dropping the $\nabla\cdot\vec{B}_0$ term. Maxwell's equations then give $\vec{B}_0\approx\nabla\psi\times\vec{E}_0$, and $\vec{E}_0\approx\nabla\psi\times\vec{B}_0$, so $(\nabla\psi)^2=1$, i.e. $\nabla\psi=\widehat{n}$, a unit vector. Get $\overline{u}=\frac{1}{16\pi}(|\vec{E}_0|^2+|\vec{B}_0|^2)=\frac{1}{8\pi}\vec{E}_0\cdot\vec{E}_0^*$ and $\overline{\vec{S}}=\frac{c}{8\pi}\vec{E}_0^*\times\vec{B}_0=\frac{c}{8\pi}(\vec{E}_0\cdot\vec{E}_0^*)\nabla\psi$. Light ray: $\vec{r}(s)$ with $\frac{d\vec{r}}{ds}=\widehat{n}$, and then get $\frac{d^2\vec{r}}{ds^2}=(\widehat{n}\cdot\frac{d}{d\vec{r}})\widehat{n}=\frac{1}{2}\nabla(\widehat{n}^2)=0$, so $\frac{d\vec{r}}{ds}$ is a constant, the rays are straight lines.

• Interference and diffraction. Consider a beam moving in the z direction, through some screen's hole in the x, y plane. $\vec{E}(\vec{r})=e^{ik_0z}\int\frac{d^3k}{(2\pi)^3}\vec{E}_ke^{i\vec{k}\cdot\vec{r}}$. Get $\vec{E}_k$ with a spread $\Delta k_x\geq 1/\Delta x$ and $\Delta k_y\geq 1/\Delta y$, leading to angular spread of the wave $\Delta\theta\sim\Delta k/k_0\sim\lambda/2\pi a$, where a is the diameter of the hole.

• Let $\psi(\vec{r})$ be the components of $\vec{E}$ or $\vec{B}$, get roughly $\psi(\vec{r})\sim\int_{\mathcal{A}}d^2a'\psi_{inc}(\vec{r}')\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$, treating each point on the aperture $\mathcal{A}$ as a point source. Show it more carefully:

In vacuum, components of $\vec{E}$ and $\vec{B}$ and $\vec{A}$ satisfy the wave equation, $(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})\psi=0$. Write $\psi=Re\psi(\vec{r})e^{-ickt}$, then $(\nabla^2+k^2)\psi(r)=0$. Solve with Green's function

$$G(\vec{r}',\vec{r})=-\frac{e^{ik|\vec{r}'-\vec{r}|}}{4\pi|\vec{r}'-\vec{r}|}+F(\vec{r}')$$

with F a solution of the laplace equation, included to get the desired BCs. So $(\nabla^2+k^2)G = \delta(\vec{r}-\vec{r}')$ and then get

$$\psi(\vec{r}) = \int_V \nabla\cdot(\psi\,\nabla G - G\,\nabla\psi)dV = -\int_{\partial V}(\psi(\vec{r}')\,\nabla' G - G\,\nabla'\psi(\vec{r}'))d\vec{a}'.$$

Take $\partial V = \mathcal{A}$, an aperture. Choices of G: Kirchoff: $G_K = -e^{ik|\vec{r}-\vec{r}'|}/4\pi|\vec{r}-\vec{r}'|$, Dirichlet: $G_D|_{r'\in\mathcal{A}} = 0$ (useful if ψ given on $\mathcal{A}$); Neumann $\widehat{n}\cdot\nabla' G_N|_{\mathcal{A}} = 0$, useful if $\nabla\psi$ given. Finding G_N or G_D is difficult in general, but it is easy for the case of an infinite plane: $G_{D,N} = G_K(\vec{r},\vec{r}') \mp G_K(\vec{r},\vec{r}'')$, where $\vec{r}''$ is the mirror of $\vec{r}'$ through the aperture, so the plane is $\vec{r}' = \vec{r}''$ and $\widehat{n}'\cdot\vec{r}' = -\widehat{n}'\cdot\vec{r}''$.

For light going through an aperture $\mathcal{A}$, and then propagating to an observer. The incident wave at $\mathcal{A}$ is $\psi_{inc}(\vec{r}')$, get (using G_D, approximating it as that above)

$$\psi(\vec{r}) = \frac{-ik}{2\pi}\int_{\mathcal{A}}\psi_{inc}(\vec{r}')\frac{e^{ik|\vec{r}'-\vec{r}|}}{|\vec{r}'-\vec{r}|}\cos\theta da',$$

where $d\vec{a}' = \widehat{z}da'$ and $\widehat{z}\cdot\nabla'|\vec{r}'-\vec{r}| = -z/|\vec{r}'-\vec{r}| = -\cos\theta$. Actually, it's complicated to really work in terms of $\vec{E}$ and $\vec{B}$, so treat scalar problem, gives good picture of the physics. For $r \gg r'$, approximate $|\vec{r}-\vec{r}'| \approx r - \vec{r}\cdot\vec{r}'/r + \frac{1}{2r^2}(\vec{r}\times\vec{r}')^2 + \ldots$. The first order term is an overall phase, unimportant. The Fraunhofer regime is when r/a is large enough to keep just the next term. The Fresnel regime is when the third term must also be kept, because ka^2/r is not small.

Example (book): Fresnel diffraction from straight edge, with screen $y < 0$, with $\psi_{inc} = \psi_0$ constant for $y \geq 0$. Observer at $\vec{r} = (0, h, D)$, get there

$$\psi \approx \frac{-ik}{2\pi}e^{\frac{ikD}{D}}\psi_0\int_0^\infty dy'\int_{-\infty}^\infty dx' e^{ik(x'^2+(y'-h)^2)/2D}$$