203a Homework 7, due Mar 13

1. Consider $\vec{A} = \hat{z}(2I/c)\ln(r_0/r)$ in cylindrical coordinates, which is the vector potential for a wire carrying current I on the $\hat{z}$ axis. An electron is ejected from $r = r_0$ with velocity $\vec{v} = v_0\hat{r}$.

   (a) Write out L in cylindrical coordinates.

   (b) Find all conserved quantities.

   (c) Find the maximum value of r reached by the electron.

2. A long solenoid of radius a is along the z axis, and attached to a disk. Inside the solenoid, $\vec{B} \approx B_0\hat{z}$, a constant. The whole thing is able to rotate freely around the z axis. The disk has a total charge Q glued to it, as a line density of charges at distance R (with $R > a$) from the z axis. Treat this question non-relativistically:

   (a) Find $\vec{A}$ outside the solenoid, choosing it to point in the $\hat{\theta}$ direction, (pointing around the $\hat{z}$ axis), so $\vec{A} = A_\theta\hat{\theta}$.

   (b) Write the Lagrangian for the system, in terms of the moment of inertia and a single coordinate, θ, the angular coordinate for rotation around the z axis, $L = L(\theta, \dot{\theta})$. Don't forget to include the coupling of the $\vec{A}$ from the part above to the charge Q.

   (c) What is the canonical momentum p_θ conjugate to θ? Since

   (d) Suppose that, initially $B \neq 0$ and $\dot{\theta} = 0$. Then, slowly, $B \to 0$. Find the $\dot{\theta}$ at later times. The apparent violation of conservation of angular momentum demonstrates that the initial fields themselves had some angular momentum (since $\vec{r} \times (\vec{E} \times \vec{B}) \neq 0$), and the total angular momentum of the device + the fields is of course conserved.

3. A particle of mass M is at rest and then it decays into three particles with masses ordered as $m_1 > m_2 > m_3$. Which particle can emerge with the largest total energy, and what is that energy? Use conservation of p_μ^{tot} and $p_i^2 = m_i^2c^2$.

4. A particle of mass m_1 and energy E_1 collides with a particle of mass m_2 that was at rest. They stick together, forming a single particle of mass M. Find M, given E_1, m_1, and m_2.

5. Compton effect: a photon of frequency ω scatters off an electron at rest. Using conservation of p_μ^{tot} with $p_\mu^{photon} = \hbar k^\mu = \hbar(\omega/c, \vec{k})$, show that the outgoing photon has ω' with

$$\frac{1}{\hbar\omega'} - \frac{1}{\hbar\omega} = \frac{2}{mc^2}\sin^2\frac{\theta}{2},$$

   where θ is the scattering angle between the incident and scattered photons.

6. Show $\partial(t', x', y', z')/\partial(t, x, y, z) = 1$ for a general Lorentz transformation. Hint: decompose the general transformation into a product of boosts and rotations.

7. Consider the following three cases for the electric and magnetic fields in some frame K (with P some constant):

i. $\vec{E} = (4P, 0, 0)$ and $\vec{B} = (0, 5P, 0)$

ii. $\vec{E} = (5P, 0, 0)$, $\vec{B} = (0, 4P, 0)$

iii. $\vec{E} = (P, 0, 0)$, $\vec{B} = (P, 2P, 0)$.

a. In which of these cases is there a frame K' where the field is purely electric? For each such case, write out the Lorentz transformation, $x^\mu = \Lambda^\mu_\nu x'^\nu$, between the frame K and the frame K' where $\vec{E}' = E'_0 \widehat{x}$ and $\vec{B}' = 0$? What is E'_0 in terms of P?

b. In which of the above cases is there a frame K' where the field is purely magnetic? For each such case, write out the Lorentz transformation $x^\mu = \Lambda^\mu_\nu x'^\nu$, between the frame K and the frame K' where $\vec{E}' = 0$ and $\vec{B}' = B'_0 \widehat{y}$? What is B'_0 in terms of P?

c. For the case or cases found in part (b), solve for the motion of a charge q particle which is at $x^\mu = 0$, with velocity $\vec{v} = 0$, in frame K. Solve for the trajectory $x'^\mu(t') = (ct', \vec{x}'(t'))$ seen in the frame K', where the field is purely magnetic.

8. Consider an infinite wire, of constant charge per length λ' that is at rest along the $\widehat{x}$ axis in the rocket frame, that is moving with velocity $\vec{v} = v\widehat{x}$ relative to the lab frame.
(a) Find ρ and $\vec{J}$ in the lab frame.

(a) Compute $\vec{E}'$ and $\vec{B}'$ in the rocket frame and transform them to the lab frame, to find $\vec{E}$ and $\vec{B}$.

(b) Find $\vec{E}$ and $\vec{B}$ directly from the ρ and $\vec{J}$ in the lab frame. Verify that they agree with those found above.

9. Two parallel wires have separation d and carry charge per length λ in the lab frame. They are at rest in the lab frame, with zero current.

(a) Calculate the force per length between the wires in a $'$ frame that moves parallel to the wires with velocity v. Do this by finding λ' and $\vec{J}'$ and using them to compute $\vec{E}'$ and $\vec{B}'$ and the associated force.

(b) Compute the same quantity as in part (a) by instead computing the force in the lab frame, and Lorentz transforming it to the lab frame. Use the fact that f^μ is a 4-vector and $d^2 r_\perp = d^2 r'_\perp$.