203a Homework 4, hints

1. Just work it out.

2. Just work it out.

3. Garg 35.2 (very similar to the above).

   Just work it out.

4. Garg 36.1.

   You guessed it.

5. Garg 36.3.

   Work it out.

6. Garg 36.6.

   As a warmup, consider a single wire at the origin, $\vec{B} = \frac{2I}{rc}\widehat{\phi}$, using cylindrical coordinates (r, ϕ, z). Then $T_{zz} = T_{rr} = -T_{\phi\phi} = \frac{1}{8\pi}(2I/rc)^2$. You can check that $\int_{\partial V} T_{ij} da^j = 0$ for any closed surface. For example, consider a cylinder around the origin, of height h and radius a. The integral of T_{zz} on the tops and bottoms cancel, since the outward normal da^j points in opposite directions. The integral of T_{rr} over the $r = a$ surface also gives zero, since $\widehat{r}$ averages to zero when integrated over all ϕ. As a more interesting cancellation, consider a region which does not enclose the wire, with r between r_1 and r_2, and ϕ between ϕ_1 and ϕ_2, and z between z_1 and z_2. The integral of T_{zz} over the top and bottom again cancel. But the integral of T_{rr} over the surfaces at fixed r_1 and r_2 do not cancel each other. And the integral of $T_{\phi\phi}$ over the surfaces at fixed ϕ_1 and ϕ_2 do not cancel each other. Nontrivially, but as expected, you can verify that all four surface contributions sum to zero. To show this, you can derive and use $\int_{\theta_1}^{\theta_2} \widehat{r}(\theta) = -\widehat{\theta}(\theta)|_{\theta_1}^{\theta_2}$. Since any volume can be built up from little bricks of the type just described, this shows that T_{ij} integrates to zero over the surface of any volume.

   Now consider two wires. So $\vec{B} = \vec{B}_{(1)} + \vec{B}_{(2)}$, where the subscripts label the wires. When you compute T_{ij} for this $\vec{B}$, you'll square the components of this $\vec{B}$. Expand out the square, you'll get $T_{ij} = T_{ij,(11)} + T_{ij,(22)} + T_{ij,(12)}$ where the (11) means involving only $\vec{B}_{(1)}$ and the (12) means involving one $\vec{B}$ from each wire. We know that $T_{ij,(11)}$ and $T_{ij,(22)}$ integrate to zero over any closed surface, we showed that above. So we only need to consider the $T_{ij,(12)}$ terms. Integrating this over a closed surface that doesn't enclose either wire must give zero. Integrating it over a closed surface that encloses either wire must give (minus) the force on that wire due to the other wire. Pick a convenient surface.

Suggestion: try a cylinder of tiny radius surrounding one wire, say wire 2. Since the radius is so small, you can approximate the $\vec{B}_{(1)}$ due to wire 1 as being constant over the surface. Work out $T_{ij,(12)}$ in this approximation and integrate it over the tiny cylinder. Verify you get the expected answer from the Lorentz force law.

7. Garg 37.1.

The idea here is to show that you can find a $\Delta T^{ij} = T^{ij}_{new} - T^{ij}_{old}$, such that $\partial_t \mathcal{P}^i + \partial_j T^{ij} = 0$ remains satisfied, with the addition to $\mathcal{P}^i$, $\Delta\mathcal{P}^i = c^{-2}\epsilon^{ijk}\partial_j F_k$. Now consider angular momentum conservation. The $\Delta\mathcal{P}^i$ leads to a $\Delta\mathcal{L}^i$ term, and the ΔT^{ij} leads to $\Delta M_{im} = \epsilon_{ijk} x_j \Delta T_{km}$ term. You can verify that these two additions almost, but don't quite sum to zero – so they screw up $\vec{L}$ conservation. In fact, the ΔT^{ij} term is antisymmetric in i, j, and $\vec{L}$ conservation requires $T_{ij} = T_{ji}$.