2/1/12 Lecture outline

⋆ Today's lecture directly follows O'Neil lecture 4 notes. See there for details.

• Last time

$$S = \int_a^b (-mc^2 d\tau - \frac{q}{c} A_\mu dx^\mu)$$

with $A^\mu = (\phi, \vec{A})$ and $dx^\mu = dt(c, \vec{v})$, led to the Euler-Lagrange equations of motion that give the relativistic Lorentz force law,

$$\frac{d}{dt}(m\vec{v}/\sqrt{1 - v^2/c^2}) = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B},$$

with

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \qquad \text{and} \qquad \vec{B} = \nabla \times \vec{A}.$$

As we saw last time, $\vec{E}$ and $\vec{B}$, and the EOM, are invariant under the gauge transformation $A_\mu \to A_\mu + \partial_\mu f$.

• Aside: P and T symmetries, $P : (ct, \vec{x}) \to (ct, -\vec{x})$ and $T : (ct, \vec{x}) \to (-ct, \vec{x})$, act on the vector potential A^μ as $P : (\phi, \vec{A}) \to (\phi, -\vec{A})$ and $T : (\phi, \vec{A}) \to (\phi, -\vec{A})$. They thus map: $P : \vec{E} \to -\vec{E},\ \vec{B} \to \vec{B}$, and $T : \vec{E} \to \vec{E},\ \vec{B} \to -\vec{B}$. These fit with $\vec{E}$ being sourced by stationary charges and $\vec{B}$ being sourced by moving charges. The RHS of the Lorentz force law transforms properly under P and T, to match that of the LHS.

• Example: $A^\mu = (\phi(\vec{x}), \vec{0})$ electrostatic. Shifting ϕ by a constant is an example of a gauge transformation.

• Example: uniform magnetic field, $\vec{B}_0 = B_0\hat{z}$, $\phi = 0$. Examples of three gauge-equivalent choices of $\vec{A}$ that all give $\nabla \times \vec{A} = \vec{B}_0$.

• Consider uniform $\vec{B}_0$, with $\vec{A} = \hat{y}B_0 x$, then

$$L = -mc^2\sqrt{1 - v^2/c^2} + \frac{q}{c}v_y B_0 x.$$

Compute conserved quantities, $E = \gamma mc^2$ and $p_y = \gamma m v_y + \frac{q}{c}B_0 x$ and $p_z = \gamma m v_z$. Helical motion with rotation in plane $\perp \vec{B}_0$: $\frac{d\vec{v}_\perp}{dt} = \Omega_c \vec{v}_\perp \times \hat{z}$, $\Omega_c = qB/\gamma mc$.

• Consider $\phi = xV/d$ and $\vec{A} = x_0 B\hat{y}$, so $\vec{E} = -\hat{x}V/d$ and $\vec{B} = B_0\hat{z}$ are constant, with $\vec{E} \cdot \vec{B} = 0$. The charge is released from rest at $x = 0$. Find minimal value of V so that charge can reach $x = d$.

$$L = -mc^2\sqrt{1 - v^2/c^2} - \frac{q}{c}B_0 x v_y + \frac{qx}{d}V.$$

With this choice of $\vec{A}$, L has y translation symmetry so $p_y = \gamma m v_y - \frac{q}{c}B_0 x$ is conserved, as is $p_z = \gamma m v_z$. Initial conditions give $p_y = p_z = 0$. The conserved energy is $E = \gamma mc^2 - qzV/d = mc^2$. Use p_y and E equations at $x = d$ to solve for V.

• Writing L requires that we introduce A^μ, with

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \qquad \text{and} \qquad \vec{B} = \nabla \times \vec{A}$$

which automatically implies that $\vec{E}$ and $\vec{B}$ satisfy two of Maxwell's equations

$$\nabla \cdot \vec{B} = 0, \qquad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

related to the non-existence of magnetic charges. If there are magnetic monopoles, it would be much harder to write down an action.

• Solenoid with cylindrical $\hat{z}$ axis symmetry and $\vec{B} = B_0\hat{z}$. Take $\vec{A} = \frac{1}{2}rB_0\hat{\theta}$ (note $\int_S \vec{B}\cdot d\vec{a} = \oint A \cdot d\vec{l}$ is satisfied as a check on $\vec{A}$.) Write Rotation symmetry: $\partial L/\partial\theta$ implies that

$$p_\theta = \frac{\partial L}{\partial\dot{\theta}} = \frac{mr^2\dot{\theta}}{\sqrt{1 - v^2/c^2}} + \frac{q}{2c}B_0 r^2.$$

This conservation law follows also from $\frac{d\vec{L}_{z,mech}}{dt} = \tau_z$.

• Let current slowly increase, replace $B_0 \to B_0(t)$. Above p_θ is still conserved (though E isn't, since $\partial L/\partial t \neq 0$). Connection with Faraday's law.

• Example with charges on disk and solenoid:

$$L = \tfrac{1}{2}I\dot{\theta}^2 + \frac{Q}{c}A_\theta(R)R\dot{\theta}.$$

$\partial L/\partial\theta = 0$ gives $P_\theta = I\dot{\theta} + \frac{Q}{c}(\Phi/2\pi)$. Suppose initially $\dot{\theta} = 0$ and $\Phi \neq 0$. Later $\Phi \to 0$ and then $\dot{\theta} \neq 0$. Q: how does this fit with conservation of angular momentum? A: as we'll discuss a bit later, there was angular momentum in the electric-magnetic field, which transferred over to mechanical angular momentum for the solenoid. Total $\vec{L}_{mech} + \vec{L}_{field}$ is conserved.