1/30/12 Lecture outline

⋆ See lecture notes for details. Continue where we left off last time.

• Generalize angular momentum, $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$, Nother conserved quantity for rotation symmetry. Generalize rotations to symmetries preserving $x^\mu x_\mu = x^{\mu'} x_{\mu'}$, i.e. rotations + boosts, $x^\mu \to \Lambda^\mu_{\nu'} x^{\nu'}$. Infinitesimal change $\delta x^\mu = \delta\Omega^{\mu\nu} x^\nu$ with $\delta\Omega_{\mu\nu} = -\delta\Omega_{\nu\mu}$. There are 6 independent generators for antisymmetric 4×4 matrices, and these are the infinitesimal generators for the 3 rotations (from $\delta\Omega_{ij} = \epsilon_{ijk} d\phi^k$) and 3 independent boosts (from $\delta\Omega_{0i} = -\delta\Omega_{i0}$, which exponentiate to give $\Lambda^{\mu'}_\nu$ for boosts).

As you know, $\vec{p} = \partial L/\partial \vec{v}$ and $E = \vec{p} \cdot \vec{v} - L$. But, as you can show, p^μ can also be written directly in terms of the action, as $p_\mu = -\frac{\partial S}{\partial x^\mu}$, where $S(x^\mu) = S_{cl}[x^\mu, x^\mu_{initial}]$ evaluating the functional for the least action path and thinking of it as a function of the final endpoint. [As an example, the non-relativistic action for a free particle traveling from $(t_a, \vec{x}_a)$ initially to $(t_b, \vec{x}_b)$ is $S = \int dt \frac{1}{2} m\vec{v}^2 = \frac{1}{2} m(\vec{x}_b - \vec{x}_a)^2/(t_b - t_a)$. So $\partial S/\partial \vec{x}_b = m\vec{v}$ and $\partial S/\partial t_b = -\frac{1}{2} m\vec{v}^2$.]

Under our rotation, the action changes by $\delta S = \sum_n \frac{\partial S}{\partial x^\mu_n} \delta x^\mu_n$, where the $\sum_n$ is over all particles. So $\delta S = -\sum_n p_\mu \delta x^\mu = \delta\Omega_{\mu\nu} \sum x^\mu p^\nu = \frac{1}{2} \delta\Omega_{\mu\nu} \sum_n (x^\mu p^\nu - x^\nu p^\mu)$. The conserved Nother quantity is $\partial S/\partial\Omega_{\mu\nu}$, so $M^{\mu\nu} = \sum (x^\mu p^\nu - p^\nu x^\mu)$ is conserved. The spatial parts are $\vec{L}$. The M^{0i} components give $\sum (t\vec{p} - E\vec{r}/c^2)$ is conserved, where $p^\mu = (E/c, \vec{)}p = \sum_n p^\mu_n$ give the total energy and momentum. Defining $\vec{R}_{CM} = \sum_i \vec{r}_i E_i/E$, the M^{0i} conservation law implies that the CM frame moves with constant velocity $\vec{V}_{CM} = \frac{d\vec{R}_{CM}}{dt} = c^2\vec{p}/E$.

• Charged particles interact with electromagnetic fields via

$$S = \int_a^b (-mc^2 d\tau - \frac{q}{c} A_\mu dx^\mu).$$

Writing $A^\mu = (\phi, \vec{A})$ and $dx^\mu = dt(c, \vec{v})$, this implies that

$$L = -mc^2\sqrt{1 - v^2/c^2} + \frac{q}{c}\vec{A} \cdot \vec{v} - q\phi.$$

• Note symmetry of δS, and thus the EOM, under $A_\mu \to A_\mu + \partial_\mu f$: **gauge invariance**; this is fundamental.

• $\vec{p} = \gamma m\vec{v} + \frac{q}{c}\vec{A}$. $\mathcal{H} = \gamma mc^2 + q\phi = c\sqrt{(\vec{p} - \frac{q}{c}\vec{A})^2 + m^2c^2} + q\phi$.

• Equations of motion: show gives Lorentz force law: $\frac{d}{dt}(m\vec{v}/\sqrt{1 - v^2/c^2}) = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B}$, with $\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$ and $\vec{B} = \nabla \times \vec{A}$.

• Note again gauge invariance: $\vec{E}$ and $\vec{B}$ are invariant under changing $A_\mu = (\phi, -\vec{A}) \to A_\mu + \partial_\mu f$, i.e. $\phi \to \phi + \frac{\partial f}{\partial t}$ and $\vec{A} \to \vec{A} - \nabla f$, for arbitrary function $f(t, \vec{x})$.