1/25/12 Lecture outline

⋆ See lecture notes for details. Continue where we left off last time.

• Last time: 4-momentum $p^\mu = (E/c, \vec{p})$. For massive particle, $p^\mu = mu^\mu$, so $p^\mu p_\mu = (mc)^2$. For a massless object (e.g. photon), we still have $p^\mu = (E, \vec{p})$ as a 4-vector. Here's a way to see that $(E, \vec{p})$ always transforms as a 4-vector. For any theory, the action S must be Lorentz invariant; this ensures that the EOM behave properly under reference frame changes. Now use the fact that energy and momentum can be related to the derivative of the action w.r.t. changes of the endpoint time and position: $L(x_b) - \dot{x}_b(\partial L/\partial \dot{x}_b) = \partial S_{cl}/\partial t_b$, and $\partial L/\partial \dot{x}_b = \partial S_{cl}/\partial x_b$, so we have $p_\mu = \partial S_{cl}/\partial x^\mu$, and the RHS is clearly a 4-vector.

• Relativistic kinematics, continued from last time. Last time we wrote the energy + momentum conservation for $M \to m_1 + m_2$ decay, with the end result that the mass m_1 particle has $E_1 = (M^2 + m_1^2 - m_2^2)c^2/2M$ (and likewise for m_2, with $1 \leftrightarrow 2$). Now demonstrate a quicker way to get the answer, from squaring $p - p_1 = p_2$.

Next example: pair creation (Jackson 11.22), scatter energetic photon against CMB photon to make an electron-positron pair. What is the minimal energy of the photon? In the CM frame the produced pair is at rest for the minimal energy. So in the frame of the CMB, the two produced particles have the same energy E and momentum p, with $E_1 + E_2 = 2E$ and $p_1 - p_2 = 2p = (E_1 - E_2)/c$. So $p_T^2 = (E_1 + E_2)^2/c^2 - (E_1 - E_2)^2/c^2 = 4E_1E_2/c^2 = (p_T')^2 = 4m^2c^2$. End result: $E_1E_2 = m^2c^4$, where $E_2 = k_BT$ for $T = 3K$.

Next example (Jackson 11.23): m_1 has $p_1 = (E_{lab}, \vec{p}_{lab})$ and m_2 is at rest in the lab frame. They collide and out comes two new particles, of mass m_3 and m_4. (a) Show that the total energy W in the CM frame is given by $W^2 = m_1^2 + m_2^2 + 2m_2E_{lab}$ [sol'n: evaluate p_T^2 in lab and CM frame], and $\vec{p}' = m_2\vec{p}_{lab}/W$ [sol'n: consider $p_1 \cdot p_T$ in lab and CM frame] (b) show that the CM frame is related to the lab frame by $\vec{\beta}_{CM} = \vec{p}_{lab}/(m_2 + E_{lab})$ and $\gamma_{CM} = (m_2 + E_{lab})/W$. (c) Show in the non-rel limit that $W \approx m_1 + m_2 + m_2p_{lab}^2/2m_1(m_1 + m_2)$ and $\vec{p}' \approx m_2\vec{p}_{lab}/(m_1 + m_2)$ and $\vec{\beta}_{cm} \approx \vec{p}_{lab}/(m_1 + m_2)$.

• $\vec{R}_{CM} = \sum_i \vec{r}_iE_i/E_T$, where $E_T = \sum_i E_i$ =constant. Then $\dot{\vec{R}}_{CM} = \sum_i \dot{\vec{r}}_iE_i/E_T = c^2\sum_i \vec{p}_i/E_T = c^2\vec{p}_T/E_T$, a constant.

• 4-acceleration $a^\mu = d^2x^\mu/d\tau^2$, satisfies $a^\mu u_\mu = 0$. Example: consider an observer who is uniformly accelerating, with acceleration $g\hat{z}$. (According Einstein's equivalence principle, this is equivalent to being in a uniform gravitational field.) So $a_\mu a^\mu = -g^2$. get $\frac{d}{dt}\gamma_v = g$, so $v_z = gt/\sqrt{1 + g^2t^2/c^2}$ and hence $z = \frac{c^2}{g}(\sqrt{1 + \frac{gt}{c^2}} - 1)$. The proper time of the accelerated observer is $\int_0^t \sqrt{1 - v^2/c^2}dt = \frac{c}{g}\sinh^{-1}(gt/c)$. At late times, $\tau \to$

$(c/g)\ln(2gt/c) \ll t$.We can relate coordinates $(\tau = \bar{t}, \bar{z})$ in the rocket frame to those in the lab via (setting c=1):

$$t = (g^{-1} + \bar{z})\sinh(g\tau) \qquad z = (g^{-1} + \bar{x})\cosh g\tau - g^{-1}.$$

(Note now $ds^2 = dt^2 - dz^2 = (1 + g\bar{z})^2 d\bar{t}^2$ which, by the equivalence principle, is related to how gravity modifies the spacetime metric from $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$ to $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$.

- Force $f^\mu = \frac{dp^\mu}{d\tau} = (\gamma c^{-1}\frac{dE}{dt}, \gamma\frac{d\vec{p}}{dt}) = (\gamma c^{-1}\vec{f}\cdot\vec{v}, \gamma\vec{f})$.
- $k^\mu = (\omega, \vec{k})$, so $e^{ik\cdot x}$ is invariant. On this, $k_\mu = i\partial_\mu$. Fits with QM, where $p^\mu = \hbar k^\mu$.

Show this gives the correct Doppler effect relations between $k^{\mu'}$ and k^μ.

- Next time: Charged particles interact with electromagnetic fields:

$$S = \int_a^b (-mc^2 d\tau - \frac{e}{c}A_\mu dx^\mu).$$