1/18/12 Lecture outline

⋆ See lecture notes for details. Continue where we left off last time, with relativity.

• Last time: Four vectors $a^\mu = (a, \vec{a})$ and $b^\mu = (b, \vec{b})$, with dot product $a \cdot b = -a_0 b_0 + \vec{a} \cdot \vec{b} \equiv a_\mu b^\mu$, where $a_\mu \equiv \eta_{\mu,\nu} a^\mu = (-a_0, \vec{b})$. Here $\eta_{\mu\nu} \equiv diag(-1,1,1,1)$ is the fixed metric of SR.

(GR replaces $\eta_{\mu\nu}$ with a dynamical metric $g_{\mu\nu}(x)$. This will be analogous to A_μ in E&M. The analog of Maxwell equations will be Einstein's equations, relating derivatives of the metric to the "charge" source of gravity: energy and momentum.)

Inertial frames are related by $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$. The dot product is preserved as long as $\eta_{\rho\sigma} = \Lambda^{\mu'}_\rho \Lambda^{\nu'}_\sigma \eta_{\mu'\nu'}$. Examples: rotate in x, y plane $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$; boost along x axis, $\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$. Consider the origin $x' = 0$ in the original frame, $x/t = v = \tanh\phi$, so $\sinh\phi = \gamma v$ and $\cosh\phi = \gamma \equiv 1/\sqrt{1 - v^2/c^2}$. Set $c = 1$ from now on.

• Continue 4-vectors a^μ, and their inner product $a \cdot b \equiv a^\mu b^\nu \eta_{\mu\nu} \equiv a^\mu b_\mu$. Two inertial frames of reference are related by (taking origins to coincide) $x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu$. The dot product is preserved as long as

$$\eta_{\rho\sigma} = \Lambda^{\mu'}_\rho \Lambda^{\nu'}_\sigma \eta_{\mu'\nu'}.$$

All Λ satisfying this form the Lorentz group. Note that all such Λ have determinant ± 1, and all those connected to the identity have determinant 1, so they have $d^4x = d^4x'$.

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Heartbeat in $'$ frame: dt', with $dx' = 0$, get $dt = \gamma dt'$, so seems to beat slower (likewise from $ds^2 = -dt^2 + d\vec{x}^2 = -dt'^2$.

Ruler in $'$ frame, length dx'. Measure both ends simultaneously in lab, with $dt = 0$, Then $dx = dx'/\gamma$, length contracted.

Two events are timelike separated if there is a frame where they happen a the same place. In that frame, $\Delta s^2 = \Delta t'^2 \equiv \Delta\tau^2$, where $\Delta\tau$ is the "proper time" between the events. In any other frame, $\Delta t = \gamma \Delta\tau$, time dilation.

For spacelike path, $\Delta s = \int ds = \int \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$.

For timelike paths, the total proper time is $\Delta\tau = \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} d\lambda$. This applies even if there is acceleration. If no acceleration, can write $\Delta\tau = \int \sqrt{1 - v^2} dt$.

Consider proper time between timelike separated events A and C. For observer 1, in the frame where they're at the same place, the proper time is $\Delta t = t_C - t_A$. For observer 2, who moves and comes back, the proper time length is $\Delta\tau_{AB'C} = \sqrt{1-v^2}\Delta\tau_{ABC} < \Delta\tau_{ABC}$. Moving twin is younger when they meet again. Non-straight path has shorter proper time. In spacetime, straight path between two events has the longest proper time.

• We saw in earlier lecture that the action is invariant under Gallean transformations provided it is $L = \frac{1}{2}m\vec{v}^2 - U(x)$, linear in v^2. To have physics be the same in all inertial frames, need the equations of motion to be covariant: if satisfied in one frame, they should be satisfied in all others, when observables are properly converted. This is achieved by having the action be a Lorentz scalar.

Free particle action $S = \int L dt$, $L = -mc^2\sqrt{1-v^2/c^2}$. Then $\vec{p} = \partial L/\partial v$ and $H = \vec{p}\cdot\vec{v} - L$ combine into $p^\mu = mu^\mu$. The EOM is then $du^\mu/d\tau = 0$ for a free particle.