2/2/11 Lecture 9 outline

• As we discussed last time, the *superficial* degree of divergence of 1PI diagrams. Consider the general form of $\Gamma^{(n)}$:

$$\Gamma^{(n)} \sim \int \prod_{i=1}^{L} \frac{d^4 k_i}{(2\pi)^4} \prod_{j=1}^{I} \frac{1}{l_j^2 - m^2 + i\epsilon}$$

For large k the integrand behaves as $\sim k^{4L-2I}$. Degree of UV divergence (superficially) is $D = 4L - 2I = 2I - 4V + 4$ (recall that $L = I - V + 1$). Suppose interaction is ϕ^p, then $pV = 2I + n$. E.g. for $\lambda\phi^4$, $p = 4$, get $D = 4 - n$.

So for $\lambda\phi^4$, $p = 4$, get $D = 4 - n$. This fits with what we found for $n = 2$, there was a quadratic divergence,

$$\Pi'(p^2) = \frac{\lambda m^2}{32\pi^2} \int_0^{\Lambda^2/m^2} \frac{u du}{u+1} = \frac{\lambda m^2}{32\pi^2} \left(\frac{\Lambda^2}{m^2} - \log(1 + \frac{\Lambda^2}{m^2}) \right).$$

i.e. $D = 2$. For $n = 4$, we get $D = 0$, which means a log divergence. For $n > 4$, we get $D < 0$, which means that there is no divergence at all (superficially, at least)! So the only two divergent cases are $n = 2$ and $n = 4$. The point will be that we can absorb these two divergent cases into corrections to the two parameters m and λ. That is the statement that the theory for $p = 4$ is renormalizable.

For $p = 6$, write $4V_4 + 6V_6 = 2I + n$, get $D = 4 - n + 2V_6$. The V_4 vertex is renormalizable, the V_6 is not. For $\lambda\phi^4$, the UV divergent terms are $n = 2, 4$. Higher n diagrams only have sub-divergences, which will be accounted for by properly treating the $n = 2$ and $n = 4$ cases. Example of a $n = 6$ diagram with a sub-divergence from the $n = 2$ diagram. Contrast $\lambda_4\phi^4$ with a $\lambda_3\phi^3$ theory (super-renormalizable) and a $\lambda_6\phi^6$ theory (non-renormalizable).

More generally, with bosons and fermions, $D = \sum_i n_i d_i + 2(IB) + 3(IF) - 4\sum_i n_i + 4$, where n_i is the number of vertices of i-th type and d_i is the number of derivatives in that interaction, and IB and IF are the numbers of internal boson and fermion lines. Then $D = -B - \frac{3}{2}F + 4 + \sum_i(\dim\mathcal{L}_i - 4)$, where B and F are the numbers of external bose and fermion lines.

• Dimensional analysis and understanding the degrees of divergence by power-counting. In $\hbar = c = 1$ units, dimensionful quantities can be written as $x \sim m^{[x]}$, which defines $[x]$, the mass dimension of x. In particular, in D space-time dimensions, we have $[S] = 0$ and $[d^D x] = -D$, so $[\mathcal{L}] = D$ so scalars have $[\phi] = (D-2)/2$ and fermions have

$[\psi] = (D-1)/2$. We see that a $\lambda_p\phi^p$ theory has $[\lambda_p] = D - p(D-2)/2$. In particular, for $D = 4$, get $[\lambda_p] = 4 - p$, showing why $p = 4$ is special, as compared with say $\lambda_3 \sim M$ and $\lambda_6 \sim M^{-2}$. Since $\Gamma^{(n)}$ has units of action, i.e. $\hbar$, it has $[\Gamma^{(n)}] = 0$. So a contribution with e.g. V_6 vertices has, on dimensional grounds, a factor of $(\lambda_6 E^2)^{V_6}$, where E is some energy scale. This reproduces the degree of UV divergence if we take $E \sim \Lambda \to \infty$. Discuss similar power counting for gravity, and for Fermi's 4-fermion weak-interaction vertex. Interpretation as low-energy effective theory with cutoff. "Non-renormalizable" theories are fine, and actually nice, in the IR, and just need some fixing up in the UV, but some UV completion.

General integrals

$$I_n(a) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+a)^n}$$

with n integer and $Im(a) > 0$ and k in Minkowski space. See

$$I_n = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{da^{n-1}} I_1(a), \qquad I_1 = \frac{-i}{16\pi^2} \int_0^{\Lambda^2} du \frac{u-a+a}{u-a}$$

where we used the solid angle $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$, which is $2\pi^2$ for $D = 4$. Get

$$I_n(a) = i\left(16\pi^2(n-1)(n-2)a^{n-2}\right)^{-1} \qquad \text{for} \quad n \geq 3.$$

Special cases

$$I_1 = \frac{i}{16\pi^2} a \ln(-a) + \ldots,$$

$$I_2 = \frac{-i}{16\pi^2} \ln(-a) + \ldots,$$

where $\ldots$ are terms involving the regulator.