1/31/11 Lecture 8 outline

• As we discussed last time, general green's functions can be built as tree-level diagrams, composed of the 1PI building blocks. This is expressed mathematically via

$$W[J] = \Gamma[\overline{\phi}] + \int d^4x J(x)\overline{\phi}(x).$$

$$\Gamma[\overline{\phi}] = W[J] - \int d^4x J(x)\overline{\phi}(x).$$

Legendre transform, like $F = E - TS$ in stat mech; get other variable via

$$\overline{\phi}(x) = \frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}, \qquad J = -\frac{\delta}{\delta\phi}\Gamma[\phi].$$

Here $\Gamma[\phi]$ is the *quantum effective action*, defined by

$$\text{1PI diagram} \equiv i\tilde{\Gamma}^{(n)}(p_1, \ldots p_n),$$

where the external propagators are amputated, and the $(2\pi)^4\delta^4(\sum_i p_i)$ is omitted, and for the 1PI propagator we define the 1PI diagram to be $-i\Pi'(p)$, and we instead define

$$i\tilde{\Gamma}^{(2)}(p,-p) = \text{1PI diagram} + i(p^2 - m^2) = i(p^2 - m^2 - \Pi'(p^2)).$$

As we saw, summing the 1PI diagrams then gives for the full propagator

$$D(p) = \frac{i}{\tilde{\Gamma}^{(2)}} = \frac{i}{p^2 - m^2 - \Pi'(p^2)},$$

so the *self energy* $\Pi'(p^2)$ is like a correction to the mass.

In position space

$$\Gamma^{(n)}(x_1, \ldots x_n) = \langle T\phi(x_1)\ldots\phi(x_n)\rangle|_{1PI}.$$

and

$$\Gamma[\phi] = \sum_{n=1}^{\infty} \frac{1}{n!}\int d^4x_1 \ldots d^4x_n \Gamma^{(n)}(x_1, \ldots x_n)\phi(x_1)\ldots\phi(x_n).$$

is called quantum effective action. As discussed last time,

$$\Gamma[\phi] = \frac{1}{\hbar}\left(S[\phi] + \mathcal{O}(\hbar)\right),$$

where the latter terms are quantum loop corrections.

Again, the point is that the quantum effects are accounted for in the quantum effective action. All quantum green's functions (and hence amplitudes) can be computed by a classical analysis (tree-diagrams), using the quantum effective action.

• Writing $\hbar\Gamma[\phi] = \int \mathcal{L}_{eff}$, the quantum effective lagrangian is of the form $\mathcal{L}_{eff} = \frac{1}{2}Z[\phi]\partial_\mu\phi\partial^\mu\phi + \ldots - V_{eff}(\phi)$, where $\ldots$ are higher derivative terms and $V_{eff}(\phi)$ is the effective potential, which determines the low-energy momentum properties of the theory.

One-loop effective potential for $\lambda\phi^4$:

$$V_{eff}^{(1)}(\phi) = i\sum_{n=1}^{\infty}\frac{1}{2n}\int\frac{d^4k}{(2\pi)^4}\left(\lambda\frac{1}{k^2-m^2+i\epsilon}\frac{\phi^2}{2}\right)^n$$
$$= \tfrac{1}{2}\int\frac{d^4k_E}{(2\pi)^4}\ln\left(1+\frac{\frac{1}{2}\lambda\phi^2}{k_E^2+m^2}\right)$$

(S. Coleman and E. Weinberg.) Symmetry factors: $1/n!$ not all the way cancelled, because of Z_n rotation symmetry, and reflection, gives $1/2n$. At each vertex, can exchange external lines, so $1/4!$ not all the way cancelled, leads to $1/2$ for each vertex. In the last expression we rotated to Euclidean space, $d^4k = id^4k_E$. Still have to explain how to handle k_E integral; we'll discuss this soon.

• Another example of a 1-loop term, the self-energy for $\lambda\phi^4$:

$$-i\Pi'(p^2) = (-i\lambda)\tfrac{1}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2} + \text{more loops.}$$

Going to Euclidean space, $d^4k = id^4k_E$,

$$\Pi'(p^2) = \tfrac{1}{2}\lambda\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2} + \text{more loops.}$$

Recall expression $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit sphere S^{D-1}. For $D=4$, get $\Omega_3 = 2\pi^2$, so

$$\Pi'(p^2) = \frac{\lambda m^2}{32\pi^2}\int_0^{\Lambda^2/m^2}\frac{udu}{u+1} = \frac{\lambda m^2}{32\pi^2}\left(\frac{\Lambda^2}{m^2} - \log(1+\frac{\Lambda^2}{m^2})\right).$$

Here Λ is a UV momentum cutoff. Result is quadratically (and also log) divergent as $\Lambda\to\infty$. The subject of renormalization is the physical interpretation of these divergences. The first thing to do is to regulate them, as we did above with a momentum cutoff. There are other ways to regulate. How one regulates is physically irrelevant. The physics is in the renormalization interpretation of the regulated results, and at the end of the day the choice of regulator doesn't matter.

• Study more generally the *superficial* degree of divergence of 1PI diagrams. Consider the general form of $\Gamma^{(n)}$:

$$\Gamma^{(n)} \sim \int \prod_{i=1}^{L} \frac{d^4 k_i}{(2\pi)^4} \prod_{j=1}^{I} \frac{1}{l_j^2 - m^2 + i\epsilon}$$

For large k the integrand behaves as $\sim k^{4L-2I}$. Degree of UV divergence (superficially) is $D = 4L - 2I = 2I - 4V + 4$ (recall that $L = I - V + 1$). Suppose interaction is ϕ^p, then $pV = 2I + n$.

E.g. for $\lambda\phi^4$, $p = 4$, get $D = 4 - n$. This fits with what we found for $n = 2$, there was a quadratic divergence, i.e. $D = 2$. For $n = 4$, we get $D = 0$, which means a log divergence. For $n > 4$, we get $D < 0$, which means that there is no divergence at all (superficially, at least)! So the only two divergent cases are $n = 2$ and $n = 4$. The point will be that we can absorb these two divergent cases into corrections to the two parameters m and λ. That is the statement that the theory for $p = 4$ is renormalizable.