1/10/11 Lecture outline

• Last time, path integral in QM. Consider QM with Hamiltonian $H(q,p)$, modified by introducing a source for q, $H \to H - f(t)q$. Replace $q \to \phi$, to make the generalization to QFT more evident. (We could also add a source for p, but don't bother doing so here.) Consider moreover replacing $H \to H(1-i\epsilon)$, with $\epsilon \to 0^+$, which has the effect of projecting on to the ground state at $t \to \pm\infty$. As mentioned last lecture, this'll be related to the $i\epsilon$ of the Feynman propagator. Consider the vacuum-to vacuum amplitude in the presence of the source,

$$\langle 0|0\rangle_J = \int [d\phi] \exp[i \int dt (L + J(t)\phi)/\hbar] \equiv Z[J(t)].$$

Once we compute $Z[J(t)]$ we can use it to compute arbitrary time-ordered expectation values. Indeed, $Z[J]$ is a generating functional[1] for time ordered expectation values of products of the $q(t)$ operators:

$$\langle 0|T\prod_{i=1}^n \phi_H(t_i)|0\rangle/\langle 0|0\rangle = Z_0^{-1}\int [d\phi]\prod_{i=1}^n \phi(t_i)\exp(iS/\hbar) = Z_0^{-1}\prod_{i=1}^n \frac{\hbar}{i}\frac{\delta}{\delta J(t)}|_{J=0}.$$

with $Z_0 = \int [d\phi]\exp(iS/\hbar)$. The time evolution $e^{-iHt/\hbar}$ is accounted for on the LHS by taking the operators in the Heisnberg picture.

Example of QM harmonic oscillator (scaling $m=1$),

$$Z[J(t)] = \int [d\phi(t)] \exp(-\frac{i}{\hbar}\int dt \left[\tfrac{1}{2}\phi(t)(\frac{d^2}{dt^2}+\omega^2)q(t) - J(t)\phi(t)\right]).$$

This is analogous to the multi-dimenensional gaussian, as discussed last time, with i replaced with the continuous label t, $\sum_i \to \int dt$ etc. and the matrix B_{ij} is replaced with the differential operator $B \to \frac{i}{2\hbar}(\frac{d^2}{dt^2}+\omega^2 - i\epsilon)$, where the $i\epsilon$ is to damp the gaussian, as mentioned above. Doing the gaussian gives a factor of $\sqrt{\det B}$ which we don't need to compute now because it'll cancel, and the exponent with the sources from completing the square, which is the term we want. That involves B^{-1}, which we can compute by Fourier transforming. In the end, we get

$$\langle 0|0\rangle_J = \exp[-\tfrac{1}{2}\hbar^{-1}\int dt dt' J(t)G(t-t')J(t')],$$

[1] Recall how functional derivatives work, e.g. $\frac{\delta}{\delta f(t)}f(t') = \delta(t-t')$.

with $G(t)$ the Green's function for the oscillator, $i(\partial_t^2 + \omega^2 i - \epsilon)G(t) = i\delta(t)$,

$$G(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \frac{ie^{-iEt/\hbar}}{E^2/\hbar^2 - \omega^2 + i\epsilon} = \frac{1}{2\omega} e^{-i\omega|t|}. \tag{1}$$

The $-i\epsilon$ here does the same thing as in the Feynman propagator: the pole at $E = \hbar\omega$ is shifted below the axis and that at $E = -\hbar\omega$ is shifted above. Equivalently, we can replace $E \to E(1 + i\epsilon)$, to tilt the integration contour below the $-\omega$ pole and above the $+\omega$ pole. Note then that $e^{-iEt/\hbar} \to e^{-iEt/\hbar} e^{Et\epsilon/\hbar}$, which projects on to the vacuum for $t \to \infty$ (the $i\epsilon$ projects on to the vacuum in the far future and also the far past).

For $t > 0$, the E contour is closed in the LHP and the residue is at $E = \hbar\omega$, while for $t < 0$ the contour is closed in the UHP, with residue at $E = -\hbar\omega$.

- On to QFT and the Klein-Gordon theory,

$$Z_0 = \int [d\phi] e^{iS/\hbar} \qquad S = \tfrac{1}{2} \int d^4x \phi(x)(-\partial^2 - m^2)\phi(x),$$

where we integrated by parts and dropped a surface term.

This is completely analogous to our QM SHO example, simply replacing $\frac{d^2}{dt^2} + \omega^2 - i\epsilon$ there with $\partial^2 + m^2 - i\epsilon$ here – again, the $i\epsilon$ is to make the oscillating gaussian integral slightly damped. I.e. we should take $S = \frac{1}{2} \int d^4x \phi(x)(-\partial^2 - m^2 + i\epsilon)\phi(x)$, with $\epsilon > 0$, and then $\epsilon \to 0^+$. Note that the operator is $B \sim -\partial^2 - m^2 + i\epsilon$, which in momentum space is $p^2 - m^2 + i\epsilon$. Looks familiar: it's the Feynman $i\epsilon$ prescription, which you understood last quarter as needed to give correct causal structure of greens functions, here comes simply from ensuring that the integrals converge! This is why the path integral automatically gives the time ordering of the products. So

$$Z_0 = \mathrm{const}(\det(-\partial^2 - \mathrm{m}^2 + \mathrm{i}\epsilon))^{-1/2}.$$

As in the SHO QM example, we can compute field theory Green's functions via the generating functional

$$Z[J(x)] = \int [d\phi] \exp(i \int d^4x [\mathcal{L} + J(x)\phi(x)]).$$

This is a functional: input function $J(x)$ and it outputs a number. Use it to compute

$$\langle 0|T \prod_{i=1}^{n} \phi(x_i)|0\rangle / \langle 0|0\rangle = Z[J]^{-1} \prod_{j=1}^{n} \left(-i \frac{\delta}{\delta J(x_i)}\right) Z[J]\big|_{J=0}.$$

E.g. for the QFT KG example, we have $B = (-i/2\hbar)(-\partial^2 - m^2 + i\epsilon)$, so $B^{-1} = 2i\hbar(-\partial^2 - m^2 + i\epsilon)^{-1}$. We then get for the generating functional

$$Z_{free}[J] = Z_0[J] = \exp(-\tfrac{1}{2}\hbar^{-1}\int d^4x d^4y J(x) D_F(x-y) J(y)), \tag{2}$$

with

$$D_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon},$$

D_F gives $i(-\partial^2 - m^2 + i\epsilon)^{-1}$.

Can use this generating function to compute free field time ordered products.. it reproduces Wick's theorem, Feynman diagrams.

As mentioned last time, S-matrix amplitudes are related (LSZ) to time-ordered products of fields. So, what we need to compute, are the Green's functions

$$G^{(n)}(x_i) \equiv \langle 0|T\prod_{i=1}^{n}\phi(x_i)|0\rangle / \langle 0|0\rangle.$$

Let's consider them in the free KG example. Find e.g. $G_0^{(2)}(x,y) = \hbar D_F(x-y)$, (where the subscript is to remind us it's the free theory), $G_0^{(4)} = G_0^{(2)}(x_1,x_2)G_0^{(2)}(x_3,x_4) +$ 2 permutations, etc.