1/5/11 Lecture outline

• Last time, path integral in QM. E.g.

$$U(x_a, x_b; T) = \langle x_b | e^{-iHT/\hbar} | x_a \rangle = \int [dx(t)] e^{iS[x(t)]/\hbar}.$$

Integral can be broken into time slices, as way to define it. E.g. free particle, get

$$U(x_b, x_a; T) = \left(\frac{2\pi i \hbar T}{m} \right)^{-1/2} \exp[im(x_b - x_a)^2/2\hbar T].$$

Note that the exponent is $e^{iS_{cl}/\hbar}$, where S_{cl} is the classical action for the classical path with these boundary conditions. (More generally, get a similar factor of $e^{iS_{cl}/\hbar}$ for interacting theories, from evaluating path integral using stationary phase.)

Plot phase of Uas a function of $x = x_b - x_a$, fixed T, Lots of oscillates. For large x, nearly constant wavelength λ, with

$$2\pi = \frac{m(x+\lambda)^2}{2\hbar T} - \frac{2m^2}{2\hbar T} \approx \frac{mx\lambda}{\hbar T} = p\lambda/\hbar.$$

Gives $p = \hbar k$.

Recover $\psi \sim e^{ipx/\hbar}$. More generally, get $p = \hbar^{-1} k$, with $p = \partial S_{cl}/\partial x_b$ (can show $p = \partial L/\partial \dot{x} = \partial S_{cl}/\partial x_b$. Can also recover $\psi \sim e^{-i\omega T}$, with $\omega = \hbar^{-1}(-\partial S_{cl}/\partial t_b)$. Agrees with $E = \hbar\omega$, since $E = p\dot{x} - L = -\partial S_{cl}/\partial t_b$.

• The same derivation leads to e.g.

$$\langle q_4, t_4 | T \widehat{q}(t_3) \widehat{q}(t_2) | q_1, t_1 \rangle = \int [dq(t)] q(t_3) q(t_2) e^{iS/\hbar},$$

where the integral is over all paths, with endpoints at (q_1, t_1) and (q_4, t_4).

A key point: the functional integral automatically accounts for time ordering! Note that the LHS above involves time ordered operators, while the RHS has a functional integral, which does not involve operators (so there is no time ordering). The fact that the time ordering comes out on the LHS is wonderful, since know that we'll need to have the time ordering for using Dyson's formula, or the LSZ formula, to compute quantum field theory amplitudes.

• The nice thing about the path integral is that it generalizes immediately to quantum fields, and for that matter to all types (scalars, fermions, gauge fields). E.g.

$$\langle \phi_b(\vec{x}, T) | e^{-iHT} | \phi_a(\vec{x}, 0) \rangle = \int [d\phi] e^{iS/\hbar} \qquad S = \int d^4x \mathcal{L}.$$

This is then used to compute Green's functions:

$$\langle\Omega|T\prod_{i=1}^{n}\phi_H(x_i)|\Omega\rangle = Z_0^{-1}\int[d\phi]\prod_{i=1}^{n}\phi(x_i)\exp(iS/\hbar),$$

with $Z_0 = \int[d\phi]\exp(iS/\hbar)$. Again, as noted above, the T ordering will be automatic.

• Now introduce **sources** for the fields as a trick to get the time order products from derivatives of a generating function (or functional).

• Consider QM with Hamiltonian $H(q,p)$, modified by introducing a source for q, $H \to H - f(t)q$. (We could also add a source for p, but don't bother doing so here.) Consider moreover replacing $H \to H(1-i\epsilon)$, with $\epsilon \to 0^+$, which has the effect of projecting on to the ground state at $t \to \pm\infty$. As mentioned last lecture, this'll be related to the $i\epsilon$ of the Feynman propagator. Consider the vacuum-to vacuum amplitude in the presence of the source,

$$\langle 0|0\rangle_f = \int[dq]\exp[i\int dt(L + f(t)q)/\hbar] \equiv Z[f(t)].$$

Once we compute $Z[f(t)]$ we can use it to compute arbitrary time-ordered expectation values. Indeed, $Z[f]$ is a generating functional[1] for time ordered expectation values of products of the $q(t)$ operators:

$$\langle 0|\prod_{j=1}^{n}Tq(t_j)|0\rangle = \prod_{j=1}^{n}\frac{1}{i}\frac{\delta}{\delta f(t_j)}Z[f]\big|_{f=0},$$

where the time evolution $e^{-iHt/\hbar}$ is accounted for on the LHS by taking the operators in the Heisnberg picture.

We'll be interested in such generating functionals, and their generalization to quantum field theory (replacing $t \to (t, \vec{x})$).

• We can explicitly evaluate the generating functional for the case of gaussian integrals, e.g. the harmonic oscillator example.

To see how, let's first consider ordinary (non functional), multi-dimensional gaussian integrals:

$$\prod_{i=1}^{N}d\phi_i\exp(-(\phi, B\phi)) = \pi^{N/2}(\det B)^{-1/2},$$

where $(\phi, B\phi) = \sum_i \phi_i(B\phi)_i$ and $(B\phi)_i = \sum_j B_{ij}\phi_j$. The integral was evaluated by changing variables in the $d\phi_i$, to the eigenvectors of the symmetric matrix B; then the integrals

[1] Recall how functional derivatives work, e.g. $\frac{\delta}{\delta f(t)}f(t') = \delta(t-t')$.

decouple into a product of simple 1-variable gaussians. As before, we'll be interested in the gaussians with an i in the exponent, which we evaluate as mentioned before,

$$\int_{-\infty}^{\infty} d\phi \exp(ia\phi^2) = \sqrt{\frac{i\pi}{a}}.$$

where we analytically continued from the case of an ordinary gaussian integral. Think of a as being complex. Then the integral converges for $\mathrm{Im}(a) > 0$, since then it's damped. To justify the above, for real a, we need the integral to be slightly damped, not just purely oscillating. To get this, take $a \to a + i\epsilon$, with $\epsilon > 0$, and then take $\epsilon \to 0^+$. So we'll replace $B \to -i(A + i\epsilon)$.

• Now discuss generating functions. First consider ordinary (non-functional) gaussian integrals. We'd like to evaluate integrals like

$$\prod_{i=1}^{N} \int d\phi_i f(\phi_i) \exp(-(\phi, B\phi))$$

for functions, like products of the ϕ_i. We can do this by computing a generating function:

$$\prod_{i=1}^{N} \int d\phi_i f(\phi_i) \exp(-B_{ij}\phi_i\phi_j) = f(\frac{\partial}{\partial J_i}) Z(J_i)\big|_{J_i=0}$$

Where

$$Z(J_i) \equiv \prod_{i=1}^{N} \int d\phi_i \exp(-B_{ij}\phi_i\phi_i + J_i\phi_i)$$

Evaluate via completing the square: the exponent is $-(\phi, B\phi) + (J, \phi) = -(\phi', B\phi') + \frac{1}{4}(J, B^{-1}J)$, where $\phi' = \phi - \frac{1}{2}B^{-1}J$. So

$$Z(J_i) = \prod_{i=1}^{N} \int d\phi_i \exp(-B_{ij}\phi_i\phi_j + J_i\phi_i) = \pi^{N/2}(\det B)^{-1/2} \exp(B_{ij}^{-1} J_i J_j/4)$$

We'll want to compute amplitudes like

$$\frac{\langle 0| \prod_i Tq(t_i)|0\rangle_{J=0}}{\langle 0|0\rangle_{J=0}}$$

and for these the $\det B$ factor above will cancel between the numerator and the denominator. This is related to the cancellation of vacuum bubble diagrams. The important piece above is the exponent with the sources.

• Now let's apply the above to compute the generating functional for the example of QM harmonic oscillator (scaling $m = 1$),

$$Z[J(t)] = \int [dq(t)] \exp(-\frac{i}{\hbar} \int dt \left[\tfrac{1}{2} q(t)(\frac{d^2}{dt^2} + \omega^2) q(t) - J(t) q(t) \right]).$$

This is analogous to the multi-dimenensional gaussian above, where i is replaced with the continuous label t, $\sum_i \to \int dt$ etc. and the matrix B_{ij} is replaced with the differential operator $B \to \frac{i}{2\hbar}(\frac{d^2}{dt^2} + \omega^2 - i\epsilon)$, where the $i\epsilon$ is to damp the gaussian, as mentioned above. Also, we replace $J \to \frac{i}{\hbar} J$ as compared with above. So doing the gaussian gives a factor of $\sqrt{\det B}$ which we don't need to compute now because it'll cancel, and the exponent with the sources from completing the square, which is the term we want. That involves B^{-1}, which we can compute by Fourier transforming. In the end, we get

$$\langle 0|0 \rangle_J = \exp[-\tfrac{1}{2}\hbar \int dt dt' J(t) G(t - t') J(t')],$$

with $G(t)$ the Green's function for the oscillator, $i(\partial_t^2 + \omega^2 i - \epsilon) G(t) = i\delta(t)$,

$$G(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \frac{i e^{-iEt/\hbar}}{E^2/\hbar^2 - \omega^2 + i\epsilon} = \frac{1}{2\omega} e^{-i\omega|t|}. \tag{1}$$

The $-i\epsilon$ here does the same thing as in the Feynman propagator: the pole at $E = \hbar\omega$ is shifted below the axis and that at $E = -\hbar\omega$ is shifted above. Equivalently, we can replace $E \to E(1 + \subset \epsilon)$, to tilt the integration contour below the $-\omega$ pole and above the $+\omega$ pole. Note then that $e^{-iEt/\hbar} \to e^{-iEt/\hbar} e^{Et\epsilon/\hbar}$, which projects on to the vacuum for $t \to \infty$ (the $i\epsilon$ projects on to the vacuum in the far future and also the far past).

For $t > 0$, the E contour is closed in the LHP and the residue is at $E = \hbar\omega$, while for $t < 0$ the contour is closed in the UHP, with residue at $E = -\hbar\omega$.

• Now that we know the generating functional, we can use it to compute time ordered expectation values via

$$\langle 0|T \prod_{i=1}^{n} \phi_H(t_i)|0\rangle / \langle 0|0\rangle = Z_0^{-1} \int [d\phi] \prod_{i=1}^{n} \phi(t_i) \exp(iS/\hbar) = Z_0^{-1} \prod_{i=1}^{n} \frac{\hbar}{i} \frac{\delta}{\delta J(t)} |_{J=0}.$$

with $Z_0 = \int [d\phi] \exp(iS/\hbar)$.