3/14/11 Lecture 19 outline

• Last time, the photon has 1PI propagator $i\Pi^{\mu\nu}(k) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(k^2)$. Summing these gives the full propagator. Writing it in Feynman gauge, get for the full propagator $-ig_{\mu\nu}/p^2(1-\Pi(p^2))$. Assuming that $\Pi(p^2)$ is regular at $p^2=0$, get pole at $p^2=0$ with residue $Z_3 \equiv (1-\Pi(0))^{-1}$.

The electron has the full propagator $S(p) = i/(\not p - m - \Sigma(p)$, where for $p$ near $m$, $S(p) = iZ_2/(\not p - m)$.

1PI vertex for electron interacting with photon, $-ie\Gamma^\mu(p', p)$. The tree-level term is $-ie\gamma^\mu$. The photon has momentum $q = p' - p$ and, for $q \to 0$, $\Gamma^\mu(p+q, p) \to Z_1^{-1}\gamma^\mu$. Can show that Lorentz and kinematic structure is such that

$$Z_2\Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2),$$

where $\sigma^{\mu\nu} = \frac{1}{2}i[\gamma^\mu, \gamma^\nu]$ and $F_i$ are “form factors.” The electron has magnetic moment $\vec{\mu} = g(e\vec{S}/2m)$, with $g = 2 + 2F_2(0)$. The diagram for $F_2(0)$ at one-loop is convergent (don’t even need to renormalize it), and yields $F_2(0) = \alpha/2\pi$. The diagram for $F_1(q^2)$ is UV, and also IR divergent at $q^2 = 0$; needs renormalization.

A W-T identity implies

$$S(p+k)(-iek_\mu)\Gamma^\mu(p+k, p)S(p) = e(S(p) - S(p+k))$$

So

$$-ik_\mu\Gamma^\mu(p_k, p) = S^{-1}(p+k) - S^{-1}(p)$$

It’s easily verified to work for the free propagators, and the W-T identity shows it’s an exact result in the full, interacting theory. Taking $p$ near on-shell and $k$ near 0, this gives $Z_1 = Z_2$; this is an important consequence of gauge invariance. As we’ll see more below, among other things, it ensures that all charged fields (e.g. the electron and the muon) couple to the gauge field with the same effective charge.

Computed 1-loop contributions, e.g.

$$\Pi(p^2) = -\frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\left(\frac{2}{\epsilon} - \gamma + \log(4\pi/\Delta)\right).$$

We’ll need to renormalize this.

Bare and renormalized fields, and counterterms. $\psi_B = Z_2^{1/2}\psi_R$, $A_B^\mu = Z_3^{1/2}A_R^\mu$, $e_B Z_2 Z_3^{1/2} = e_R Z_1$. $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$.

$$\mathcal{L}_R = -\frac{1}{4}F_{R\mu\nu}F_R^{\mu\nu} + \bar{\psi}_R(i\partial\!\!\!/ - e_R A\!\!\!/_R - m_R)\psi_R,$$

$$\mathcal{L}_{ct} = -\frac{1}{4}\delta_3(F_{R\mu\nu})^2 + \bar{\psi}_R(i\delta_2\partial\!\!\!/ - \delta_1 e_R A\!\!\!/_R - \delta_m)\psi_R.$$

Where $\delta_1 = Z_1 - 1$, $\delta_2 = Z_2 - 1$, $\delta_3 = Z_3 - 1$, and $\delta_m = Z_2 m_0 - m$.

• Continue. In particular, the counter-term contributes to $i\Pi^{\mu\nu}$ as $\delta\Pi = -(Z_3 - 1)$. So, to one loop, we get

$$\Pi(p^2) = -\frac{\alpha}{\pi}\epsilon^{-1}\frac{2}{3} + (Z_3 - 1)^{(1)} + \text{finite}.$$

in MS, choose $Z_3$ to cancel the $1/\epsilon$ term only, so $\delta_3 \equiv Z_3 - 1 = -\frac{\alpha}{\pi}\epsilon^{-1}\frac{2}{3}$.

We'll soon note that $e_{phys} = \sqrt{Z_3}e_B$, or better $\alpha = e_{phys}^2/4\pi = Z_3\mu^{-\epsilon}\alpha_B$. Write this as $\alpha_B = \alpha\mu^\epsilon Z_\alpha$, where

$$Z_\alpha \equiv Z_3^{-1} \equiv 1 + \sum_k a_k(\alpha)\epsilon^{-k}.$$

In particular, we found above that $a_1 = 2\alpha/3\pi$ to one-loop order.

Just like what we did in $\lambda\phi^4$, use the fact that $\alpha_B$ is independent of $\mu$ to get

$$0 = \epsilon\alpha Z_3^{-1} + \beta(\alpha,\epsilon)Z_3^{-1} + \beta(\alpha,\epsilon)\alpha\frac{d}{d\alpha}Z_3^{-1}.$$

where $\beta(\alpha,\epsilon) = d\alpha/d\ln\mu$. To have a smooth $\epsilon \to 0$ limit, we need

$$\beta(\alpha,\epsilon) = -\epsilon\alpha + \beta(\alpha),$$

$$\beta(\alpha) = \alpha^2\frac{da_1}{d\alpha}.$$

Using the above result for $a_1$, we get finally

$$\beta(\alpha) = \frac{d\alpha}{d\ln\mu} = \frac{2\alpha^2}{3\pi} + \text{higher loops}.$$

This is the promised beta function of QED. It's positive, as in $\lambda\phi^4$, and every other theory except non-Abelian gauge theories. Its sign is again related to charge screening, so the effective charge is small at long distances (IR free) and blows up at short distances (the Landau pole), as we discussed before. Integrate 1-loop beta function:

$$\alpha^{-1}(\mu) = -\frac{2}{3\pi}\ln(\frac{\mu}{\Lambda}).$$

Makes sense only for $\mu < \Lambda$, i.e. in the IR. $\Lambda$ is a UV cutoff. Get $\alpha \to \infty$ as $\mu \to \Lambda$; this is the Landau pole. Looks bad, but we'll see the the energy scale where it blows up is so fantastically large that we don't need to worry (something new should fix it in the UV, e.g. grand unification can do the job). It does not run to zero in the IR, because there are no massless charged particles. It runs toward zero until it gets to the energy scale of the lightest charged particle, $m_e = 0.5 MeV$, and then it stops running. So $137 = \frac{3}{3\pi}\ln(\Lambda/m_e)$. Gives $\Lambda = m_e \exp(137\pi)$, which too huge to worry about the apparent Landau pole there. (Other charged particles will bring the scale of $\Lambda$ down to $\Lambda = m_e \exp(137\pi/N_f)$ where $N_f$ is the effective number of charged particles, but it's still huge.)

• For the 1-loop correction to the 1PI 2-point function for the electron, the counter-terms plus the virtual photon correction to the electron propagator diagram gives

$$-i\Sigma_2(p) = -i\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dx \frac{\Gamma(2-\frac{1}{2}d)}{((1-x)m^2 - x\mu^2 - (x(1-x)p^2)^{2-d/2}}((4-\epsilon)m - (2-\epsilon)\not{p})$$
$$+ i(\not{p}\delta_2 - \delta_m),$$

(where $\mu$ is a small photon mass, temporarily put in by hand to cure an IR divergence). In MS this gives to 1-loop

$$\delta_2 = -\frac{\alpha}{2\pi\epsilon}, \qquad \delta_m = \frac{2\alpha}{\pi\epsilon}.$$

• For the 1-loop correction to the vertex, the diagram with a virtual photon, and the counter-term, contribute to the form factor $F_1(q^2)$. (Again, the $F_2(q^2)$ loop correction is finite.) In MS, get $\delta_1 = -\alpha/2\pi\epsilon$. Note $\delta_1 = \delta_2$.

• As we said already, gauge invariance requires $Z_1 = Z_2$, so $\delta_1 = \delta_2$ must hold exactly. Then the counterterm pieces have the same gauge invariance. (This is a special case of a more general Ward identity, stating $\Gamma_\mu(p,p) = -\partial_{p^\mu}\Sigma(p)$.)

So $e_{phys} = \sqrt{Z_3}e_B$. This shows that renormalized charge is same for all species. E.g. electron and muon and anti-proton all have exactly the same effective charge.

• QED vs QED. In QED, we have gauge invariance $\psi \to e^{ief(x)}\psi$, local $U(1)$ transformations. Generalize to local $SU(N_c)$ gauge transformations: $\psi \to U^f(x)\psi = \exp(igT^a f_a(x))\psi$, where $T^a$ are traceless, Hermitian $N_c \times N_c$ matrices ($a = 1\dots N_c^2 - 1$), and $\psi$ is a $N_c$ column vector. Gauge conserved color charge. Need covariant derivatives, $\partial_\mu \to D_\mu = \partial_\mu - igA_\mu^a T^a$, i.e. introduce gauge fields, "gluons". The $T_a$ matrices do not commute, $[T^a, T^b] = if_{abc}T^c$: the group is "non-Abelian." (They are normalized b $\mathrm{Tr}T^aT^b = \frac{1}{2}\delta^{ab}$, e.g. for $SU(2)$, $T^a = \sigma^a$, the Pauli matrices.) The effect of this is that the $A^a_\mu$ kinetic terms are more complicated. The physics of this is that the gluons carry color charge (unlike the photon, which carries no electric charge).

Gauge transformation: $D_\mu\psi \to D^f_\mu U^f\psi = U^f D_\mu\psi$, i.e. $D_\mu \to UD_\mu U^{-1}$, i.e. $A^f_\mu = UA^f_\mu U^{-1} - ig^{-1}(\partial_\mu U)U^{-1}$.

Field strength: $[D_\mu, D_\nu] = -igF^{\mu\nu}$, i.e. $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A^\mu, A^\nu]$, i.e. $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$.

Lagrangian

$$\mathcal{L}_{gaugekinetic} = -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a}, \qquad \mathcal{L}_{ferm} = \bar{\psi}(i\not{D} - m)\psi.$$

Some parts are similar to QED, e.g. the gauge field propagator is $iD^{ab}_{\mu\nu} = \frac{-i\delta^{ab}}{k^2+i\epsilon}(g_{\mu\nu} - (\xi-1)k^\mu k^\nu/k^2)$. Some differences from QED: since gluons are charged, get 3 and 4 gluon diagrams, as seen from expanding $\mathcal{L}_{gaugekinetic}$. These yield added contributions to 1-loop correction to gluon propagator. (We also have to gauge fix and consequently add Faddeev Popov ghosts, e.g. gauge fixing by $G(A) = \partial^\mu A_\mu - \omega(x)$ leads to the FP determinant $\det(\frac{\delta G(A^\alpha)}{\delta\alpha}) \sim \det(\partial^\mu D_\mu)$ and then $\mathcal{L}_{g.f.+ghost} = -\frac{1}{2\xi}(\partial_\mu A^\mu) - c^\dagger\partial^\mu D_\mu c$. Ghosts only appear in closed loops, where the contribution has a minus sign since they're anticommuting fields.)

• Recall $e^+e^- \to \mu^+\mu^-$ at tree level in QED, with total cross section $\sigma = \frac{4\pi\alpha^2}{3s}\sqrt{1-\frac{m^2_\mu}{s}}(1+\frac{m^2_\mu}{2s}) \approx \frac{4\pi\alpha^2}{3s}$ at high energy. The total cross section for $e^+e^- \to$ hadrons at high energy is the same, up to a factor of $3\sum_i Q_i^2$, where $Q_i$ accounts for the electric charge of the quarks and 3 accounts for their color. This gave an experimental verification of 3 colors.

• Renormalization.

Consider gauge boson 1PI loop contribution, $i(p^2g^{\mu\nu} - p^\mu p^\nu)\delta^{ab}\Pi(p^2)$. Fermions contribute

$$\Pi(p^2) \supset -\frac{g^2}{16\pi^2}\frac{4}{3}N_f T_2(r)\Gamma(2-\tfrac{1}{2}d) + \dots .$$

Now add 3 diagrams: two with internal gluons, and one with internal ghost. Each is separately quadratically divergent and would induce a gauge boson mass. But these problems cancel in the sum. The upshot of the sum is

$$\Pi(p^2) \supset -\frac{g^2}{16\pi^2}(-(\frac{13}{6} - \tfrac{1}{2}\xi))C(G)\Gamma(2-\tfrac{1}{2}d) + \dots .$$

To compute the beta function, must account for loop diagrams involving the fermion vertex. It's somewhat involved (see Peskin). But there is a nice way to determine it from the gauge field propagator in what's known as background field gauge, where one includes a classical background for the field and gauge fixes around that.

Get finally

$$\beta(\alpha) = \frac{\alpha^2}{6\pi} \left(-11N_c + 2N_f\right).$$

(More generally, replace $N_c \to C_2(G)$ and $2N_f \to 4n_f T_2(r)$.) The flavors contribute positively, as in QED. But the colors contribute negatively: they anti-screen charges! So the beta function can be negative, if $11N_c > 2N_f$. This is asymptotic freedom. Integrating the 1-loop result gives

$$\alpha(\mu)^{-1} = \frac{(11N_c - 2N_f)}{6\pi} \ln(\frac{\mu}{\Lambda}).$$

To have $\alpha > 0$, we need $\mu > \Lambda$ (opposite from QED). Note $\alpha(\mu \to \infty) \to 0$, weak in UV = asymptotic freedom. Explains successes of parton model (quarks) for high energy scattering. For QCD, $N_c = 3$, and $N_f = 6$. For energies below the top and bottom mass, use $N_f^{eff} = 4$. Observe e.g. $\alpha(100GeV) \sim 0.1$, so $\Lambda \sim 200MeV$.

On the other hand, $\alpha \to \infty$ for $\mu \to \Lambda$: forces are strong in IR, below scale $\Lambda$. Can explain confinement of quarks (there is a million dollar prize, waiting to be collected, if you prove it in detail)!

• Phases of QCD.