3/9/11 Lecture 18 outline

• Last time, propagators for free, spin 1/2 fermions:

$$\frac{i}{\not k - m + i\epsilon},$$

and gauge field

$$D_{\mu\nu} = \frac{-i}{k^2}[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \xi \frac{k_\mu k_\nu}{k^2}]$$

Popular choices: $\xi = 1$ is Feynman propagator, $\xi = 0$ is Landau gauge propagator. Physics is ξ independent (result of gauge invariance, which yields Ward-Takahashi identities). Let's choose to use Feynman gauge.)

• Recall QED Feynman rules, e.g. vertex: $-ie\gamma^\mu$.

• The photon has 1PI propagator $i\Pi^{\mu\nu}(k) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(k^2)$. Summing these gives the full propagator. Writing it in Feynman gauge, get for the full propagator $-ig_{\mu\nu}/p^2(1 - \Pi(p^2))$. Assuming that $\Pi(p^2)$ is regular at $p^2 = 0$, get pole at $p^2 = 0$ with residue $Z_3 \equiv (1 - \Pi(0))^{-1}$.

The electron has the full propagator $S(p) = i/(\not p - m - \Sigma(p)$, where for p near m, $S(p) = iZ_2/(\not p - m)$. The 1PI interaction vertex (with electron having incoming momentum p (and outgoing momentum $p + k$) and photon having incoming momentum k) is $-ie\Gamma^\mu(p + k, p)$, where for $k \to 0$, $\Gamma^\mu(p + k, p) \to Z_1^{-1}\gamma^\mu$.

The W-T identity is

$$S(p + k)(-iek_\mu)\Gamma^\mu(p + k, p)S(p) = e(S(p) - S(p + k))$$

So

$$-ik_\mu \Gamma^\mu(p_k, p) = S^{-1}(p + k) - S^{-1}(p)$$

It's easily verified to work for the free propagators, and the W-T identity shows it's an exact result in the full, interacting theory. Taking p near on-shell and k near 0, this gives $Z_1 = Z_2$; this is an important consequence of gauge invariance. As we'll see more below, among other things, it ensures that e.g. the electron and the muon couple to the gauge field with the same effective charge.

• Compute the correction to the photon propagator from a virtual electron/positron loop:

$$i\Pi^{\mu\nu}(q) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} tr\left(\gamma^\mu \frac{i}{\not k - m}\gamma^\nu \frac{i}{\not k + \not q - m}\right).$$

Combine denominators using Feynman parameter

$$\frac{1}{(k^2-m^2)((k+q)^2-m^2} = \int_0^1 dx \frac{1}{(\ell^2+x(1-x)q^2-m^2)^2}$$

with $\ell = k + xq$. Go to Euclidean space and do integrals using our previous tables of integrals in dim-reg to find

$$\Pi(p^2) = -\frac{8e^2}{(4\pi)^{d/2}}\Gamma(2-\tfrac{1}{2}d)\int_0^1 dx x(1-x)\Delta^{\frac{1}{2}d-2},$$

with $\Delta = m^2 - x(1-x)p^2$. Evaluating for $d = 4-\epsilon$,

$$\Pi(p^2) = -\frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\left(\frac{2}{\epsilon}-\gamma+\log(4\pi/\Delta)\right).$$

We'll need to renormalize this.

• Let's note some other interesting things about the finite part of $\Pi(p^2)$. $\Pi(p^2)$ has a branch cut starting at $p^2 = 4m^2$, and its imaginary part above and below the cut have

$$Im(\Pi(p^2 \pm i\epsilon) = \mp\frac{\alpha}{3}\sqrt{1-\frac{4m^2}{p^2}}(1+\frac{2m^2}{p^2}),$$

which is related by the optical theorem to the total cross section for creating an on-shell fermion-antifermion pair,

$$\frac{d\sigma}{d\Omega} = \frac{|\vec{p}|}{32\pi^2 s^{3/2}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2.$$

• Likewise, can compute the contribution of a virtual photon to the full electron propagator

$$S(p) = \frac{i}{\not{p}-m-\Sigma(p)+i\epsilon},$$

where $-i\Sigma$ is the 1PI contribution to the propagator. E.g. to 1 loop get

$$-i\Sigma(p^2) = (-ie)^2\int\frac{d^4k}{(2\pi)^4}\frac{-ig_{\mu\nu}}{k^2}\gamma^\mu\frac{i}{\not{p}-\not{k}-m}\gamma^\nu.$$

The function $S(p)$ has a pole at the physical mass, $m_{phys} = m+\Sigma(0)$, so the constant part of Σ shifts the mass. The $\sim\not{p}$ part of Σ renormalizes the residue of $S(p)$. The residue is iZ_2. Again, we can add counterterms to shift these and preserve a renormalization condition.

• 1PI vertex for electron interacting with photon, $-ie\Gamma^\mu(p',p)$. The tree-level term is $-ie\gamma^\mu$. The photon has momentum $q = p' - p$. Can show that Lorentz and kinematic structure is such that

$$Z_2\Gamma^\mu(p',p) = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2),$$

where $\sigma^{\mu\nu} = \frac{1}{2}i[\gamma^\mu, \gamma^\nu]$ and F_i are "form factors." The electron has magnetic moment $\vec{\mu} = g(e\vec{S}/2m)$, with $g = 2 + 2F_2(0)$. The diagram for $F_2(0)$ at one-loop is convergent (don't even need to renormalize it), and yields $F_2(0) = \alpha/2\pi$. The diagram for $F_1(q^2)$ is UV, and also IR divergent at $q^2 = 0$; needs renormalization. Define $\Gamma^\mu(q^2 = 0) = Z_1^{-1}\gamma^\mu$. The W.T. identity shows $F_1(0) = 1$.

• We now renormalize. Bare and renormalized fields, and counterterms. $\psi_B = Z_2^{1/2}\psi_R$, $A_B^\mu = Z_3^{1/2}A_R^\mu$, $e_B Z_2 Z_3^{1/2} = e_R Z_1$. $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$.

$$\mathcal{L}_R = -\frac{1}{4}F_{R\mu\nu}F_R^{\mu\nu} + \bar{\psi}_R(i\partial\!\!\!/ - e_R A\!\!\!/_R - m_R)\psi_R,$$

$$\mathcal{L}_{ct} = -\frac{1}{4}\delta_3(F_{R\mu\nu})^2 + \bar{\psi}_R(i\delta_2\partial\!\!\!/ - \delta_1 e_R A\!\!\!/_R - \delta_m)\psi_R.$$

Where $\delta_1 = Z_1 - 1$, $\delta_2 = Z_2 - 1$, $\delta_3 = Z_3 - 1$, and $\delta_m = Z_2 m_0 - m$.

In particular, the counter-term contributes to $i\Pi^{\mu\nu}$ as $\delta\Pi = -(Z_3 - 1)$. So, to one loop, we get

$$\Pi(p^2) = -\frac{\alpha}{\pi}\epsilon^{-1}\frac{2}{3} + (Z_3 - 1)^{(1)} + \text{finite}.$$

in MS, choose Z_3 to cancel the $1/\epsilon$ term only, so $Z_3 - 1 = -\frac{\alpha}{\pi}\epsilon^{-1}\frac{2}{3}$.

We'll soon note that $e_{phys} = \sqrt{Z_3}e_B$, or better $\alpha = e_{phys}^2/4\pi = Z_3\mu^{-\epsilon}\alpha_B$. Write this as $\alpha_B = \alpha\mu^\epsilon Z_\alpha$, where

$$Z_\alpha \equiv Z_3^{-1} \equiv 1 + \sum_k a_k(\alpha)\epsilon^{-k}.$$

In particular, we found above that $a_1 = 2\alpha/3\pi$ to one-loop order.