3/7/11 Lecture 17 outline

• New topic: quantum field theory for fields with spin, in particular spin 1/2 fermions and spin 1 gauge fields, for example for QED [1]. Path integral of same, general form, but need to understand some new issues with the integrations. Consider fermions first, where the functional integral is over grassmann valued fields. As you saw in a HW set, grassmann number integrals work like $\int d\theta(A+B\theta)=B$. Complex θ, θ^*, $\int d\theta^* d\theta \exp(-\theta^* b\theta)=b$. $\prod_i \int d\theta_i^* d\theta_i \exp(-\theta_i^* B_{ij}\theta_j)=\det B$. $\prod_i \int d\theta_i^* d\theta_i \exp(-\theta_i^* B_{ij}\theta_j)\theta_k\theta_l^* = (B^{-1})_{kl}\det B$.

• We can introduce sources for the fields:

$$
\begin{aligned}
Z[\bar\eta_i,\eta_i] &= \int d\bar\theta_i d\theta_i \exp(i(A_{ij}\bar\theta_i\theta_j + \bar\eta_i\theta_i + \bar\theta_i\eta_i]) \\
&= \int d\bar\theta_i d\theta_i (1+i(\bar\theta, A\theta))(1+i\bar\eta\theta)(1+i\bar\theta\eta), \\
&= -i\det A \exp(-i\bar\eta_i A^{-1}_{ij}\eta_j).
\end{aligned}
$$

• Generalize to functional integrals over fermionic fields;

$$
\begin{aligned}
Z[\bar\eta,\eta] &= \int [d\bar\psi][d\psi]\exp(i\int d^4x[\bar\psi(i\partial\!\!\!/ - m)\psi + \bar\eta\psi + \bar\psi\eta] \\
&= Z_0 \exp[-\int d^4x d^4y \bar\eta(x) S_F(x-y)\eta(y).
\end{aligned}
$$

where

$$
S_F[x-y] = i(i\partial\!\!\!/ - m)^{-1} = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(x-y)}}{k\!\!\!/ - m + i\epsilon}.
$$

Get e.g.

$$
\langle 0|T\psi(x)\bar\psi(y)|0\rangle = Z_0^{-1}(-i\frac{\delta}{\delta\bar\eta(x)})(i\frac{\delta}{\delta\eta(y)})Z[\eta,\bar\eta]|_{\eta,\bar\eta=0} = S_F(x-y).
$$

This gives the Feynman rules for fermions that you saw last quarter.

• For fermions, the $\det B$ is in the numerator, whereas for scalars it's in the denominator. The functional integral gives e^{iW}. So the sign of the contribution to W is opposite for closed scalar vs fermion loops: every closed fermion loop gets an extra -1 factor. (This relative minus sign is put to good use with supersymmetry!)

• Functional integral for gauge fields. Important point: gauge invariance. Write $A = A_\mu dx^\mu$. Recall gauge symmetry $A \to A^\alpha = A + d\alpha(x)$, with $\psi \to e^{-ie\alpha(x)}\psi$. Redundancy

[1] Fields with higher spin, e.g. the spin 2 metric, whose quanta are gravitons, can also be treated with the path integral, though they are non-renormalizable so a UV cutoff is required. Additional physics (e.g. string theory) can give a UV completion of the theory above the cutoff.

in description, can only observe gauge invariant quantities. Need to replace $\partial_\mu \to D_\mu \equiv \partial_\mu + ieA_\mu$. Then $D^\alpha_\mu \psi^\alpha = e^{-ie\alpha} D_\mu \psi$ transforms nicely, with just an overall phase, and $\bar\psi D_\mu \psi$ is gauge invariant. So the Dirac lagrangian, $\bar\psi(i\not{D} - m)\psi$ is gauge invariant. In functional integral, will have $\int [dA] \exp(iS)$. Integration measure must be gauge invariant, implies it gets a factor of gauge orbit volume. Would like to integrate only over a slice of inequivalent gauge fields, without integrating over the gauge orbits. Need to do this, since otherwise there is no well defined B^{-1}. Recall $S = \int d^4x[-\frac{1}{4}F^2_{\mu\nu}] = \frac{1}{2}\int d^4k A_\mu(x)(\partial^2 g^{\mu\nu} - \partial^\mu\partial^\nu)A_\nu(x)$. Note action vanishes if $\tilde{A}_\mu(k) = k_\mu \alpha(k)$. Gauge invariance. $A^T_\mu = P_{\mu\nu}A^\nu$, $P_{\mu\nu} = g_{\mu\nu} - \partial_\mu\partial_\nu/\partial^2$. $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}A^T_\mu \partial^2 g^{\mu\nu} A^T_\nu$. Can't invert kinetic terms uniquely to find Green's function. We need to fix the gauge.

The functional integral should be over $\int [dA^\mu]/(GE)$, where we divide by the volume of the gauge equivalent orbits. The gauge equivalent orbits are associated with gauge transformations $\alpha(x)$, e.g. $A_\mu \to A_\mu + \partial_\mu \alpha(x)$ in the Abelian case. We want to do the functional integral over A^μ, dividing out by the $\alpha(x)$.

(Here are some details: Do this via

$$1 = \int [d\alpha(x)] \delta(G(A^\alpha)) \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right) = \Delta \int [d\alpha] \delta(G(A^\alpha)),$$

where $G(A) = 0$ is some gauge fixing condition, e.g. Lorentz gauge, $G(A) = \partial_\mu A^\mu$ and

$$\Delta = \det\left(\frac{\delta G(A^\alpha)}{\delta\alpha}\right)_{G=0}.$$

Δ is the Faddeev-Popov determinant. Write the functional integral as (using the gauge invariance of measure and action)

$$\int [d\alpha][dA] \Delta \delta(G[A]) \exp(iS[A]).$$

Have factored out the integral over the group volume. We can then just easily divide out by $[d\alpha]$, just cross it out. What's left is the gauge fixing delta function, and appropriate determinant factor.

Take e.g. $G = \partial^\mu A_\mu - f(x)$ for some function $f(x)$. Then $\Delta \sim \det(\partial^2)$ is a constant. Get

$$e^{iW} = N \int (dA) e^{iS} \delta(\partial^\mu A_\mu - f) = N \int [dA][df] e^{iS} \delta(\partial^\mu A_\mu - f) G(f) = N \int [dA] e^{iS} G(\partial A),$$

for arbitrary functional G. Choose $G(f) = \exp(-\frac{1}{2}i\xi^{-1}\int d^4xf^2)$, for some real number ξ. Get

$$e^{iW} = N\int [dA]\exp(iS - \tfrac{1}{2}\xi^{-1}\int d^4x(\partial^\mu A_\mu)^2).$$

Then get for the propagator

$$D_{\mu\nu} = \frac{-i}{k^2}[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \xi\frac{k_\mu k_\nu}{k^2}].$$

Popular choices: $\xi = 1$ is Feynman propagator, $\xi = 0$ is Landau gauge propagator. Physics is ξ independent (result of gauge invariance, which yields Ward-Takahashi identities). Let's choose to use Feynman gauge.)

• Gauge invariance shows up in the amplitudes by what's know as the Ward-Takahashi identities. Consider a green's function $\langle 0|Tj^\mu(x)\prod_i \Phi(x_i)|0\rangle$, where j^μ is the conserved current and $\Phi(x_i)$ are other fields (they could be fermions). Much as you saw in a HW exercise, using the functional integral it is seen (by going through the symmetry transformation change of variables a-la Noether's procedure) that current conservation holds up to $\delta(x - x_i)$ contact terms. For example,

$$i\partial_\mu\langle 0|Tj^\mu(x)\psi(x_1)\overline{\psi}(x_2)|0\rangle = ie(\delta(x - x_2) - \delta(x - x_1))\langle 0|T\psi(x_1)\overline{\psi}(x_2)|0\rangle.$$

In momentum space,

$$-ik_\mu\mathcal{M}^\mu(k,p,q) = -ie\mathcal{M}_0(p, q - k) + ie\mathcal{M}_{\prime}(p + k, q).$$

Amplitudes with more external states are similar, with a sum over all external states weighted by their charge. When we go to S-matrix elements using the LSZ procedure, the terms on the RHS vanish when we amputate the external legs and go on-shell, so current conservation is indeed satisfied in S-matrix elements.