3/2/11 Lecture 16 outline

- Last time:

$$\tilde{\Gamma}_B^{(n)}(p_1,\ldots p_n;\lambda_B,m_B,\epsilon)=Z_\phi^{-n/2}\tilde{\Gamma}_R^{(n)}(p_1,\ldots p_n;\lambda_R,m_R,\mu,\epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point $\mu$ while he LHS does not. This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with $\mu$. Take $d/d\ln\mu$ of both sides, and use $d\Gamma_B/d\mu=0$ gives

$$\left(\frac{\partial}{\partial\ln\mu}+\beta(\lambda_R)\frac{\partial}{\partial\lambda_R}+\gamma_m\frac{\partial}{\partial\ln m_R}-n\gamma\right)\tilde{\Gamma}_R^{(n)}(p_1,\ldots p_n;\lambda_R,m_R,\mu)=0$$

with

$$\beta(\lambda)\equiv\frac{d}{d\ln\mu}\lambda_R$$

$$\gamma=\tfrac{1}{2}\frac{d}{d\ln\mu}\ln Z_\phi$$

$$\gamma_m=\frac{d\ln m_R}{d\ln\mu}.$$

E.g. for $\lambda\phi^4$, recall $\mathcal{L}_B=\mathcal{L}_R+\mathcal{L}_{c.t.}$ and $\phi_B\equiv Z_\phi^{1/2}\phi_R$, so we had $\mathcal{L}_{c.t.}=\ldots-\delta\lambda\mu^\epsilon\phi^4/4!$ where $\delta\lambda\mu^\epsilon\equiv\lambda_B Z_\phi^2-\lambda\mu^\epsilon$, which we'll rewrite as

$$\lambda_B=\mu^\epsilon Z_\phi^{-2}(\lambda+\delta_\lambda)\equiv\mu^\epsilon\lambda Z_\lambda$$

where

$$Z_\lambda\equiv Z_\phi^{-2}(1+\frac{\delta_\lambda}{\lambda})\equiv 1+\sum_{k>0}a_k(\lambda)\epsilon^{-k}.$$

The bare parameter $\lambda_B$ is independent of $\mu$, whereas $\lambda$ depends on $\mu$, such that the above relation holds. Take $d/d\ln\mu$ of both sides,

$$0=(\epsilon\lambda+\beta(\lambda,\epsilon))Z_\lambda+\beta(\lambda,\epsilon)\lambda\frac{dZ_\lambda}{d\lambda}.$$

This equation must hold as a function of $\epsilon$. Now $Z_\lambda=1+\epsilon^{negative}$, and $dZ_\lambda/d\lambda=\epsilon^{negative}$. On the other hand, $\beta(\lambda,\epsilon)=d\lambda_R/d\ln\mu$ is non-singular as $\epsilon\to 0$, so $\beta(\lambda,\epsilon)=\beta(\lambda)+\sum_{n>0}\beta_n\epsilon^n$. Plugging back into the above equation then gives

$$\beta(\lambda,\epsilon)=-\epsilon\lambda+\beta(\lambda)$$

$$\beta(\lambda) = \lambda^2 \frac{da_1}{d\lambda}$$

$$\lambda^2 \frac{da_{k+1}}{d\lambda} = \beta(\lambda) \frac{d}{d\lambda}(\lambda a_k),$$

where the first comes from $\epsilon^n$, the second from $\epsilon^0$, and the third from $\epsilon^{-k}$, with $n, k > 0$.

So the beta function is determined entirely from $a_1$. The $a_{k>1}$ are also entirely determined by $a_1$. In $k$-th order in perturbation theory, the leading pole goes like $1/\epsilon^k$.

Recall that we found for $\lambda\phi^4$, in MS where we found to 1-loop

$$\delta_m = \frac{\lambda m^2}{16\pi^2} \frac{1}{\epsilon}, \qquad \delta_\lambda = \frac{3\lambda^2}{16\pi^2} \frac{1}{\epsilon}, \qquad \delta_Z = 0.$$

So we find $a_1(\lambda) = +3\lambda/16\pi^2$ to one loop. This gives

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3).$$

- Let's now discuss, similarly with the dim reg discussion of last time,

$$\gamma_\phi(\lambda, \epsilon) = \tfrac{1}{2} \frac{d}{d \ln \mu} \ln Z_\phi$$

where

$$Z_\phi = 1 + \sum_k Z_\phi^{-k}(\lambda)\epsilon^{-k}.$$

So

$$\gamma_\phi(\lambda, \epsilon) = \tfrac{1}{2}\beta(\lambda, \epsilon) \frac{d}{d\lambda} \ln Z_\phi.$$

Using $\beta(\lambda, \epsilon) = -\epsilon\lambda + \beta(\lambda)$, we get

$$\gamma_\phi = -\tfrac{1}{2}\lambda \frac{d}{d\lambda} Z_\phi^{(1)}.$$

We similarly have $m_B^2 = (m^2 + \delta_{m^2})Z_\phi^{-1} \equiv Z_m m^2$ and

$$\gamma_m(\lambda) = \tfrac{1}{2} \frac{d \ln m^2}{d \ln \mu} = -\tfrac{1}{2} \frac{d \ln Z_m}{d \ln \mu} = -\tfrac{1}{2}\beta \frac{d \ln Z_m}{d\lambda} = \tfrac{1}{2}\lambda \frac{dZ_m^{(1)}}{d\lambda}$$

where $Z_m^{(1)}$ means the coefficient of $1/\epsilon$. In all these cases, only the coefficient of $1/\epsilon$ matters.

In particular, for $\lambda\phi^4$ we have

$$\gamma_m(\lambda) = \tfrac{1}{2}\lambda \frac{dZ_m^{(1)}}{d\lambda} = \tfrac{1}{2} \frac{\lambda}{16\pi^2} - \frac{5}{12} \frac{\lambda^2}{6(16\pi^2)^2} + \ldots$$

where $Z_m^{(1)}$ means the coefficient of $1/\epsilon$ and ... are higher orders in perturbation theory, and

$$\gamma_\phi = -\tfrac{1}{2}\lambda \frac{d}{d\lambda} Z_\phi^{(1)} = \frac{1}{12}\frac{\lambda^2}{(16\pi^2)^2} + \ldots$$

For any gauge invariant field $\phi$, we always have $\gamma_\phi \geq 0$, where $\gamma_\phi = 0$ iff it is a free field. This follows from the spectral decomposition result that $Z \leq 1$.

The anomalous dimension $\gamma_\phi$ is an additional quantum correction to the classical scaling dimension of the field: $\Delta(\mathcal{O}) = \Delta_{cl}(\mathcal{O}) + \gamma_{\mathcal{O}}$, e.g. here we find to 1-loop that $\Delta(\phi) = 1 + \frac{1}{12}\frac{\lambda^2}{(16\pi^2)^2}$.

• Let's discuss the RG equation in another way – the "Wilsonian" RG picture. Suppose that we break the path integral $\int [d\phi(k)]$ up into the "fast" modes, with $|k_E| > M$, and the "slow" ones with $|k_E| < M$, for some cutoff $M$. First do the integral over the fast modes, to get a low-energy effective theory lagrangian for the slow modes. This effective lagrangian has an effective coupling $\lambda$. Physics at the end of the day doesn't care about where we put $M$, but the effective coupling $\lambda$ must vary with $M$ to compensate for the fact that ultimately physics is $M$ independent. Likewise, if we change $M \to M'$, we need to rescale $\phi' = Z_\phi^{-1/2}(M', M)\phi$. The condition that physics is independent of $M$ is

$$\left(\frac{\partial}{\partial \log M} + \beta(\lambda)\frac{\partial}{\partial\lambda} + n\gamma(\lambda)\right)\widetilde{\Gamma}_R^{(n)}(p_1, \ldots p_n, M, \lambda) = 0,$$

with

$$\beta = \frac{d\lambda}{d\log M} \qquad \gamma = \tfrac{1}{2}\frac{d\ln Z_\phi}{d\ln M},$$

this is an alternative, equivalent interpretation of the same RG equations seen before.

• Integrating the 1-loop beta function that we found for $\lambda\phi^4$ theory gives

$$\lambda = \lambda_0\left(1 - \frac{3}{16\pi^3}\lambda_0 \ln(\mu/\mu_0)\right)^{-1}.$$

Or we can write the effective $\lambda(p) = \lambda(1 - (3\lambda/16\pi^2)\log(p/M))^{-1}$.

• All physical parameters, masses couplings etc satisfy the RG eqn:

$$\mathcal{D}P \equiv \left(\frac{\partial}{\partial \ln\mu} + \beta(\lambda)\frac{\partial}{\partial\lambda} + \gamma_m\frac{\partial}{\partial \ln m}\right)P(\lambda, m, \mu) = 0.$$

• Note: $\beta > 0$ means the coupling is small in the IR, and large in the UV. Such theories are "not asymptotically free" or are "IR free." Most theories are like this, e.g. $\lambda\phi^4$ (e.g. the Higgs coupling), QED, Yukawa interactions.

• QED: one loop beta function, $\beta(e) = e^3/12\pi^2$, leads to $\alpha_{eff}(\mu)^{-1} = \alpha_0^{-1} - \frac{1}{6\pi}\log(\mu/\mu_0)$. Again, positive beta function. Discuss the interpretation. Picture for QED of vacuum polarization, screening the bare charge.

4d theories without non-abelian gauge fields all have $\beta > 0$. They then need a cutoff to define them in the UV, and tend to flow to free theories in the IR.

• QCD has $\beta < 0$: the coupling is small in the UV, and large in the IR. Such theories are "asymptotically free;" only non-Abelian gauge theories, like QCD, are like that. Means vacuum anti-screens charges. QCD: one loop beta function $\beta(g) = -Cg^3/2$, leads to $g^{-2}(\mu) = g_0^{-2} + C\log(\mu/\mu_0)$.

• Pictures of RG flows. Briefly outline GUT idea and unification of the running couplings.