2/28/11 Lecture 15 outline

• Appetizer: consider $\mathcal{L} = \frac{1}{2}\partial\phi^2 - \frac{\lambda}{4!}\phi^4$, this class' favorite interacting theory, but now without the mass term. This theory is classically scale invariant, since there is no classical mass scale, so we might imagine that the theory is invariant under rescaling $x_\mu \to Cx_\mu$ for general constant parameter $C$. But this classical scale invariance is broken at the quantum level! The quantum theory (i.e. loops) requires renormalization, which introduces a scale, e.g. the scale $\mu$ in dim reg, where $\lambda_{old} = \lambda_{new}\mu^{4-D} = \lambda_{new}\mu^{-\epsilon}$. On the other hand, this scale is sort-of fake. The renormalization group (RG) is how we account for that.

•Let's consider more generally

$$\tilde{\Gamma}_B^{(n)}(p_1, \dots p_n; \lambda_B, m_B, \epsilon) = Z_\phi^{-n/2}\tilde{\Gamma}_R^{(n)}(p_1, \dots p_n; \lambda_R, m_R, \mu, \epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point $\mu$ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with $\mu$.

Take $d/d\ln\mu$ of both sides, and use $d\Gamma_B/d\mu = 0$. This gives

$$\left(\frac{\partial}{\partial \ln\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} + \gamma_m m_R \frac{\partial}{\partial \ln m_R} - n\gamma\right)\tilde{\Gamma}_R^{(n)}(p_1, \dots p_n; \lambda_R, m_R, \mu) = 0$$

Here

$$\beta(\lambda) \equiv \frac{d}{d\ln\mu}\lambda_R$$

$$\gamma = \tfrac{1}{2}\frac{d}{d\ln\mu}\ln Z_\phi$$

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu}.$$

This is the RG equation. Various variants, depending on subtraction procedure (scheme). For mass dependent scheme, this gives the original Gell-Mann Low equations, where $\beta$ and $\gamma$ depend on the physical mass. The Callan-Symanzik equation replaces $\partial/\partial\ln\mu$ with $\partial/\partial\ln m$, giving the change as the physical mass is varied. It's often better to use a mass-independent scheme, like MS (or $\overline{MS}$, where we had introduced the scale $M$ in replacing, via appropriate counterterms, $(\frac{2}{\epsilon} - \gamma + \log(4\pi/m^2) \to \log(M^2/m^2))$, where $m$ appears as just another coupling constant. In any case, the RG equation can be integrated, to relate the renormalized Greens functions at different scales $\mu$ and $\mu'$.

• Understand what $\beta$ and $\gamma$ mean: the bare quantities are some function of the renormalized ones and epsilon. E.g. for $\lambda\phi^4$, recall $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$ and $\phi_B \equiv Z_\phi^{1/2}\phi_R$, so we had $\mathcal{L}_{c.t.} = \ldots - \delta\lambda\mu^\epsilon\phi^4/4!$ where $\delta\lambda\mu^\epsilon \equiv \lambda_B Z_\phi^2 - \lambda\mu^\epsilon$, which we'll rewrite as

$$\lambda_B = \mu^\epsilon Z_\phi^{-2}(\lambda + \delta_\lambda) \equiv \mu^\epsilon \lambda Z_\lambda$$

where

$$Z_\lambda \equiv Z_\phi^{-2}(1 + \frac{\delta_\lambda}{\lambda}) \equiv 1 + \sum_{k>0} a_k(\lambda)\epsilon^{-k}.$$

The bare parameter $\lambda_B$ is independent of $\mu$, whereas $\lambda$ depends on $\mu$, such that the above relation holds. Take $d/d\ln\mu$ of both sides,

$$0 = (\epsilon\lambda + \beta(\lambda,\epsilon))Z_\lambda + \beta(\lambda,\epsilon)\lambda\frac{dZ_\lambda}{d\lambda}.$$

This equation must hold as a function of $\epsilon$. Now $Z_\lambda = 1+\epsilon^{negative}$, and $dZ_\lambda/d\lambda = \epsilon^{negative}$. On the other hand, $\beta(\lambda,\epsilon) = d\lambda_R/d\ln\mu$ is non-singular as $\epsilon \to 0$, so $\beta(\lambda,\epsilon) = \beta(\lambda) + \sum_{n>0}\beta_n\epsilon^n$. Plugging back into the above equation then gives

$$\beta(\lambda,\epsilon) = -\epsilon\lambda + \beta(\lambda)$$

$$\beta(\lambda) = \lambda^2\frac{da_1}{d\lambda}$$

$$\lambda^2\frac{da_{k+1}}{d\lambda} = \beta(\lambda)\frac{d}{d\lambda}(\lambda a_k),$$

where the first comes from $\epsilon^n$, the second from $\epsilon^0$, and the third from $\epsilon^{-k}$, with $n, k > 0$.

So the beta function is determined entirely from $a_1$. The $a_{k>1}$ are also entirely determined by $a_1$. In $k$-th order in perturbation theory, the leading pole goes like $1/\epsilon^k$.

Recall that we found for $\lambda\phi^4$, in MS where we found to 1-loop

$$\delta_m = \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_Z = 0.$$

So we find $a_1(\lambda) = +3\lambda/16\pi^2$ to one loop. This gives

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3).$$