2/16/11 Lecture 13 outline

• Recall last time: renormalized and bare Greens functions. Recall that $\Phi_B \equiv Z_\phi^{1/2}\phi$, and the definition of the 1PI Green's functions $\widetilde{\Gamma}^{(n)}$, and in particular that they have all n external propagators amputated. It then follows that

$$\tilde{\Gamma}_B^{(n)}(p_1, \ldots p_n; \lambda_B, m_B, \epsilon) = Z_\phi^{-n/2}\tilde{\Gamma}_R^{(n)}(p_1, \ldots p_n; \lambda_R, m_R, \mu, \epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point μ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with μ. Rewrite above as

$$Z_\phi^{n/2}\tilde{\Gamma}_B^{(n)}(p_1, \ldots p_n; \lambda_B, m_B, \epsilon) = \tilde{\Gamma}_R^{(n)}(p_1, \ldots p_n; \lambda_R, m_R, \mu, \epsilon).$$

Now the RHS is finite, so the LHS must be too. So we can take $\epsilon \to 0$ without a problem.

• Before getting into the renormalization group, let's take a little detour. Recall that

$$\int d^4x e^{ipx}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle = \frac{i}{p^2 - m^2 - \Pi'(p^2) + i\epsilon}.$$

Here $|\Omega\rangle$ is the full, interacting vacuum and ϕ are the full (Heisenberg picture) operators. Now insert a complete set of states,

$$\mathbf{1} = |\Omega\rangle\langle\Omega| + \sum_\lambda \int \frac{d^3p}{(2\pi)^2}\frac{1}{2E_p(\lambda)}|\lambda_p\rangle\langle\lambda_p|$$

where λ are all eigenstates of the full H, and λ_p is a boosted version, to give an eigenstate of $\vec{P}$, with spatial momentum $\vec{p}$. Now use $\langle\Omega|\phi(x)|\lambda_p\rangle = \langle\Omega|\phi(0)|\lambda_0\rangle e^{-ipx}$ (where $p^0 = E_p \equiv \sqrt{|\vec{p}|^2 + m_\lambda^2}$) and replace $\int \frac{d^3\vec{p}}{(2\pi)^3 2E_p} \to \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m_\lambda^2 + i\epsilon}$ to get

$$\langle\Omega\phi(x)\phi(0)|\Omega\rangle = \sum_\lambda \int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2 - m_\lambda^2 + i\epsilon}e^{-ipx}|\langle\Omega|\phi(0)|\lambda_0\rangle|^2.$$

So

$$\int d^4x e^{ipx}\langle\Omega|T\phi(x)\phi(0)|\Omega\rangle = \int_0^\infty \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon},$$

where

$$\rho(M^2) = \sum_\lambda 2\pi\delta(M^2 - m_\lambda^2)|\langle\Omega|\phi(0)|\lambda\rangle|^2 > 0$$

is the Kallen-Lehmann spectral density. Find $\rho(M^2) = 2\pi\delta(M^2 - m^2)Z$ for $M^2 \ll 4m^2$. For M^2 slightly below $4m^2$ there are new delta functions, at the bound states. Starting at $4m^2$, $\rho(M^2)$ is some positive function. This implies that

$$\frac{i}{p^2 - m^2 - \Pi'(p^2) + i\epsilon} = \frac{iZ}{p^2 - m^2 + i\epsilon} + \int_{\sim 4m^2}^{\infty} \frac{dM^2}{2\pi}\rho(M^2)\frac{i}{p^2 - M^2 + i\epsilon}.$$

The LHS has a simple pole, with residue iZ, at $p^2 = m^2$. Here $Z = |\langle\lambda_0|\phi(0)|\Omega\rangle|^2$ is the probability for $\phi(0)$ to create the lowest energy 1-particle state from the vacuum. Then there can be a few more simple poles, for p^2 slightly below $4m^2$.

Starting at $p^2 = 4m^2$, there is a branch cut, corresponding to producing two more more free particles. Note $\mathcal{M}(s) = \mathcal{M}(s^*)^*$ implies that the real part of $\mathcal{M}$ is continuous across the cut, but the imaginary part can be discontinuous: $Im\mathcal{M}(s + i\epsilon) = -Im\mathcal{M}(s - i\epsilon)$. We'll return to this shortly.

The above equality, back in position space and taking $\partial/\partial t$, leads to the equal time commutators, $[\phi(\vec{x}, t), \dot{\phi}(\vec{y}, t)] = i\delta^{(3)}(\vec{x} - \vec{y})$, matching the coefficient of the delta function on the two sides of the resulting equation gives

$$1 = Z + \int_{\sim 4m^2}^{\infty} \frac{dM^2}{2\pi}\rho(M^2) \geq Z.$$

Implies that $0 \leq Z \leq 1$, with $Z = 1$ iff the theory is a free field theory. Intuitively reasonable, since Z essentially gives the probability of ϕ to create a 1-particle asymptotic in state, given that it can also create other things. Recall what we found before,

$$\delta_Z^{(2)} = -\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\epsilon},$$

so negative (for $\epsilon > 0$).

• Recall LSZ (Lehmann, Symanzik, Zimmermann '55) from last quarter, now noting that there are Z factors. Let's just state the result: the S-matrix element for m incoming particles and n outgoing ones is given by

$$\langle \mathbf{p_1} \dots \mathbf{p_n}|S|\mathbf{k_1} \dots \mathbf{k_m}\rangle = \lim_{o.s} \prod_{i=1}^{n}(p_i^2 - m_i^2)Z_i^{-1/2} \prod_{j=1}^{m}(k_j^2 - m_j^2)Z_j^{-1/2}\tilde{G}^{n+m}(-p_i, k_i).$$

Here $\tilde{G}^{n+m}$ is the full $n + m$ point Green's function, including disconnected diagrams etc. The limit is where we take the external particles on shell. In this limit, the $p_i^2 - m_i^2$ and $k_j^2 - m_j^2$ prefactors all go to zero. These zeros kill everything on the RHS except for the connected contributions to $\tilde{G}$. Accounting for the fact that we amputate the external propagators, which go like $iZ_i(p_i^2 - m_i^2)^{-1}$, the above becomes

$$\langle \mathbf{p_1} \dots \mathbf{p_n} | S | \mathbf{k_1} \dots \mathbf{k_m} \rangle = Z^{(n+m)/2} \tilde{G}^{n+m}_{amp,conn,B}(-p_i, k_i) = \tilde{G}^{n+m}_{amp,conn,R}(-p_i, k_j)$$

Good: the physical S-matrix elements are computed from the renormalized Greens functions, which we take to be finite in our renormalization procedure.