2/14/11 ♡ Lecture 12 outline

• Recall last time we found

$$\Pi'(p^2) = -\frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \log\frac{m^2}{4\pi\mu^2} + 1 - \gamma\right) + \ldots + \delta m - p^2\delta Z$$

where ... are higher loop and $\lambda_{old} = \lambda_{new}\mu^{4-D}$ with λ_{new} dimensionless. Also,

$$\widetilde{\Gamma}^{(4)} = -\lambda\hbar^{-1} + \frac{\lambda^2}{32\pi^2}\left(3\frac{2}{\epsilon} - 3\gamma + 3\log\frac{4\pi\mu^2}{m^2} + A_1(s) + A_1(t) + A_1(u)\right) + \mathcal{O} - \delta\lambda + \ldots$$

with

$$A_1(s) = 2 - \sqrt{1 - \frac{4m^2}{s}}\log\left(\frac{\sqrt{1-\frac{4m^2}{s}}+1}{\sqrt{1-\frac{4m^2}{s}}-1}\right).$$

Recall $\mathcal{L}_B = \mathcal{L}_{phys} + \mathcal{L}_{c.t.}$, with $\phi_B \equiv Z_\phi^{1/2}\phi$, and

$$\mathcal{L}_{c.t.} = \tfrac{1}{2}(Z_\phi - 1)\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}(m_B^2 Z_\phi - m^2)\phi^2 - (\lambda_B Z_\phi^2 - \lambda\mu^\epsilon)\frac{1}{4!}\phi^4.$$

Define $\delta_Z \equiv Z_\phi - 1$, $\delta_m = m_B^2 Z_\phi - m^2$, $\delta_\lambda\mu^\epsilon = \lambda_B Z_\phi^2 - \lambda\mu^\epsilon$. There are extra diagram contributions for these corrections: there is a line (like the propagator) with an insertion of the counterterm, which gives a factor of $i(p^2\delta_Z - \delta_m)$. There is a new vertex with a factor of $-i\delta_\lambda$.

Among other things, the counter terms must be chosen to cancel the divergences, so

$$\delta_m = \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon} + \text{finite} + \mathcal{O}(\lambda^3).$$

$$\delta_\lambda = 3\frac{\lambda^2}{16\pi^2}\frac{1}{\epsilon} + \text{finite} + \mathcal{O}(\lambda^4).$$

To one loop, $\delta_Z = 0+$(finite), because $\Pi'(p^2)$ is independent of p^2.

What to do about the finite parts is a choice that we can make, called our renormalization prescription. We have to define what we're calling the physical mass and coupling. The physics will be independent of our particular choice, and different choices have different calculational advantages or disadvantages. We'll discuss three choices: (i) on shell; (ii) minimal subtraction (MS); (iii) $\overline{MS}$.

• On shell renormalization scheme. Here, we define what we mean by the mass to be the pole of the full propagator (sum of all connected diagrams), $D(p) = i/\tilde{\Gamma}^{(2)}$, and to define the physical field so that the residue of the pole is i. This means

$$\Pi'(m^2) = 0, \qquad \frac{d\Pi'}{dp^2}|_{p^2=m^2} = 0, \qquad \tilde{\Gamma}^{(4)}|_{s=4m^2} = -\lambda$$

where the last condition is our definition of physical λ. With this choice, we have

$$\delta_m = +\frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \log\frac{m^2}{4\pi\mu^2} + 1 - \gamma\right)$$

to this order, and so, to this order they combine to give

$$\Pi'(p^2) = 0.$$

We also have $\delta Z = 0$ and $\delta\lambda$ is such that now

$$\tilde{\Gamma}^{(4)} = -\lambda + \frac{\lambda^2}{32\pi^2}\left(A_1(s) + A_1(t) + A_1(u) - A_1(4m^2) - 2A_1(0)\right).$$

More generally, we can consider the "on shell" renormalization scheme, defined by imposing

$$\Pi'(m^2) = 0, \qquad \frac{d\Pi'}{dp^2}|_{p^2=m^2} = 0, \qquad \tilde{\Gamma}^{(4)}|_{s=\mu} = -\lambda$$

Above we took $\mu = 4m^2$. We could also change the renormalization point μ.

• Now mention two other renormalization schemes, which have an advantage in actual perturbative calculations in that they are *mass independent* (to be illustrated below). In minimal subtraction (MS) we choose the counterterms to remove the $1/\epsilon$ poles, and nothing else. A variant is $\overline{MS}$, where one replaces

$$\frac{\Gamma(2-\frac{1}{2}D)}{(4\pi)^{D/2}(m^2)^{2-\frac{1}{2}D}} = \frac{1}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \log(4\pi/m^2)\right)$$

with

$$\frac{1}{16\pi^2}\log(M^2/m^2),$$

for some arbitrary mass parameter M. (The advantage is that it gets rid of annoying finite constants like γ and other derivatives of the gamma function, which otherwise proliferate at each higher loop order.) The apparent freedom to define things many different ways always cancels out at the end of the day, when one relates to physical observables. Different choices have different benefits along the way.

• Let's consider $\lambda\phi^4$ in MS. To one loop, we have

$$\delta_m = \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_\lambda = \frac{3\lambda^2}{16\pi^2}\frac{1}{\epsilon}, \qquad \delta_Z = 0.$$

Now consider the propagator to two loops. Diagram 1 is a one-loop diagram with the 1-loop $\delta\lambda$ counterterm at the vertex. Diagram 2 is a one-loop diagram with the 1-loop δ_m

counterterm on the internal propagator. Diagram 3 is a two-loop diagram which looks like a double-scoop of the 1-loop diagrams. Diagram 4 is a line which cuts through a circle (see your HW). Diagram 5 has no loops, but an insertion of the 2-loop δ_m and δ_Z counter terms. Let's consider the pole terms in the diagrams. Diagram 1 requires no new computation: we can obtain it from the previous 1-loop contribution to $-i\Pi'$ by simply replacing there $\lambda \to \delta\lambda$. This gives

$$-i\Pi'_{diag\ 1} = i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{3}{2}\left(\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2} + \frac{1}{\epsilon} - \frac{\gamma}{\epsilon}\right) + \mathcal{O}(\epsilon^0)$$

Diagram 2 has 2 propagators in the loop, with the 1-loop δ_m insertion, which gives

$$-i\Pi'_{diag\ 2} = i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{1}{2}\left(\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\ln\frac{m^2}{4\pi\mu^2} - \frac{\gamma}{\epsilon}\right) + \mathcal{O}(\epsilon^0)$$

Diagram 3 contributes

$$-i\Pi'_{diag\ 3} = \frac{1}{4}(-i\lambda)^2\mu^{2\epsilon}\int\frac{d^Dk}{(2\pi)^D}\frac{i}{k^2-m^2}\int\frac{d^Dq}{(2\pi)^D}\left(\frac{i}{q^2-m^2}\right)^2,$$

where q is the integral over the lower loop, which has two propagators. This gives

$$-i\frac{\lambda^2}{(16\pi^2)^2}m^2\frac{1}{2}\left(\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\ln\frac{m^2}{4\pi\mu^2} + \frac{1}{\epsilon} - \frac{2\gamma}{\epsilon}\right) + \mathcal{O}(\epsilon^0)$$

Diagram 4 gives

$$i\frac{\lambda^2}{(16\pi^2)^2}\left(-\frac{m^2}{\epsilon^2} + \frac{1}{\epsilon}\left(m^2\ln\frac{m^2}{4\pi\mu^2} + \frac{1}{12}p^2 + (\gamma - \frac{3}{2}m^2)\right)\right)$$

Diagram 5 are the two-loop counterterms, $i\delta_Z^{(2)}p^2 - i\delta_m^{(2)}$. We should then take for the 2-loop contributions to the counterterms

$$\delta m^{(2)} = \frac{\lambda^2}{(16\pi^2)^2}\left(\frac{2}{\epsilon^2} - \frac{1}{2\epsilon}\right)m^2,$$

$$\delta_Z^{(2)} = -\frac{\lambda^2}{12(16\pi^2)^2}\frac{1}{\epsilon}.$$

The terms involving $\ln m^2/4\pi\mu^2$ all cancel. This happens for all loops. MS is a mass independent scheme, in that $\delta\lambda$, δZ, and $\delta m/m^2$ are independent of m and μ.

• Renormalized and bare Greens functions. Recall that $\Phi_B \equiv Z_\phi^{1/2}\phi$, and the definition of the 1PI Green's functions $\widetilde{\Gamma}^{(n)}$, and in particular that they have all n external propagators amputated. It then follows that

$$\tilde{\Gamma}_B^{(n)}(p_1,\ldots p_n;\lambda_B,m_B,\epsilon) = Z_\phi^{-n/2}\tilde{\Gamma}_R^{(n)}(p_1,\ldots p_n;\lambda_R,m_R,\mu,\epsilon).$$

For fixed physics, the LHS is some fixed quantity. The RHS depends on the renormalization point μ and the scheme. The LHS does not! This leads to what is known as the renormalization group equations, which state how the renormalized quantities must vary with μ. Rewrite above as

$$Z_\phi^{n/2}\tilde{\Gamma}_B^{(n)}(p_1,\ldots p_n;\lambda_B,m_B,\epsilon) = \tilde{\Gamma}_R^{(n)}(p_1,\ldots p_n;\lambda_R,m_R,\mu,\epsilon).$$

Now the RHS is finite, so the LHS must be too. So we can take $\epsilon \to 0$ without a problem.