2/9/11 Lecture 11 outline

• Last time, using dim reg ($D = 4 - \epsilon$) we found the 1-loop self energy

$$\Pi'(p^2)^{(1)} = -\frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \log\frac{m^2}{4\pi\mu^2} + 1 - \gamma\right)$$

where the scale μ entered via $\lambda_{old} = \lambda_{new}\mu^{4-D}$ with λ_{new} dimensionless. We also found

$$\tilde{\Gamma}^{(4)} = -\lambda\hbar^{-1} + (-i\lambda)^2(F(s) + F(t) + F(u)) + O(\hbar),$$

where

$$F(s_E) = -\tfrac{1}{2}\int \frac{d^4k_E}{(2\pi)^4}\int_0^1 dx \frac{1}{(k_E^2 + m^2 + x(1-x)s_E)^2}.$$

Where $s_E = p_E^2 = -s$. Evaluate the k integral our dimreg integrals. Expanding around $D = 4 - \epsilon$, gives

$$F(s_E) = -\frac{1}{32\pi^2}\int_0^1 dx\left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log(m^2 + x(1-x)s_E)\right).$$

So

$$\widetilde{\Gamma}^{(4)} = -\lambda\hbar^{-1} + \frac{\lambda^2}{32\pi^2}\left(3\frac{2}{\epsilon} - 3\gamma + 3\log\frac{4\pi\mu^2}{m^2} + A_1(s) + A_1(t) + A_1(u)\right) + \mathcal{O}(\hbar).$$

Here

$$A_1(s_E) = -\int_0^1 dx \log(1 + x(1-x)\frac{s_E}{m^2})$$

The integral is evaluated using

$$\int_0^1 dx \log(1 + \frac{4}{a}x(1-x)) = -2 + \sqrt{1+a}\log\left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1}\right) \qquad a > 0.$$

So again there is a $1/\epsilon$ term plus finite terms. (The finite terms have interesting behavior at $s, t, u = 4m^2$, which we'll discuss soon as being related to intermediate channel particles going on-shell.)

• Renormalization. The input to the functional integral is the "bare" lagrangian. It is not physically observable, because we observe quantities like mass, charge, etc. with all the quantum corrections included. Write the largrangian for the bare fields as:

$$\mathcal{L}_B = \tfrac{1}{2}\partial_\mu\phi_B\partial^\mu\phi_B - \tfrac{1}{2}m_B^2\phi_B^2 - \lambda_B\frac{1}{4!}\phi_B^4.$$

The bare field is related to the physical one by $\phi_B \equiv Z_\phi^{1/2}\phi$. We can view this as

$$\mathcal{L}_B = \mathcal{L}_{phys} + \mathcal{L}_{c.t.}$$

where

$$\mathcal{L}_{phys} = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2 - \lambda\mu^\epsilon\frac{1}{4!}\phi^4$$

involves the physical field, mass, coupling constant. What's left are the counterterms:

$$\mathcal{L}_{c.t.} = \tfrac{1}{2}(Z_\phi - 1)\partial_\mu\phi\partial^\mu\phi - \tfrac{1}{2}(m_B^2 Z_\phi - m^2)\phi^2 - (\lambda_B Z_\phi^2 - \lambda\mu^\epsilon)\frac{1}{4!}\phi^4.$$

Define $\delta_Z \equiv Z_\phi - 1$, $\delta_m = m_B^2 Z_\phi - m^2$, $\delta_\lambda\mu^\epsilon = \lambda_B Z_\phi^2 - \lambda\mu^\epsilon$. There are extra diagram contributions for these corrections.

There is a line (like the propagator) with an insertion of the counterterm, which gives a factor of $i(p^2\delta_Z - \delta_m)$. There is a new vertex with a factor of $-i\delta_\lambda$. These new diagrams count as having one loop factor (one factor of $\hbar$).

• Among other things, these corrections cancel the divergences. E.g. δm *adds* to Π', so pick the additive contribution to cancel the divergence in Π'; likewise, $\delta\lambda$ adds to effective λ obtained from $\widetilde{\Gamma}^{(4)}$, so

$$\delta_m = \frac{\lambda m^2}{16\pi^2}\frac{1}{\epsilon} + \text{finite} + \mathcal{O}(\lambda^3).$$

$$\delta_\lambda = 3\frac{\lambda^2}{16\pi^2}\frac{1}{\epsilon} + \text{finite} + \mathcal{O}(\lambda^4).$$

To one loop, $\delta_Z = 0+$(finite), because $\Pi'(p^2)$ is independent of p^2.

Amazing and non-trivial fact: we can cancel every divergence in $\lambda\phi^4$, just by using δZ, δm^2, and $\delta\lambda$. Contrast this with $\lambda_6\phi^6$, where more and more counterterms are required, e.g. the 1-loop contribution to $\widetilde{\Gamma}^{(8)}$ requires a $\delta\lambda_8\phi^8$ counterterm, and it's never ending. Renormalizable vs non-renormalizable theories.

• Renormalizability: all divergences cancelled by counter terms of the same form as original $\mathcal{L}$. This would not be the case for e.g. $\lambda\phi^6$. Even for $\lambda\phi^4$, it is quite non-trivial. For example, in doing 2 loops, there could have been some term from one loop diagrams, with counter terms, leading to $\frac{1}{\epsilon}\ln p^2$, which could not be cancelled by a counterterm in our lagrangian. Sometimes individual diagrams indeed behave like that. But the coefficients of all such terms sum to zero.

• What to do about the finite parts is a choice that we can make, called our renormalization prescription. We have to define what we're calling the physical mass and coupling.

The physics will be independent of our particular choice, and different choices have different calculational advantages or disadvantages. We'll discuss three choices: (i) on shell; (ii) minimal subtraction (MS); (iii) $\overline{MS}$.

• On shell renormalization scheme. Here, we define what we mean by the mass to be the pole of the full propagator (sum of all connected diagrams), $D(p) = i/\tilde{\Gamma}^{(2)}$, and to define the physical field so that the residue of the pole is i. This means

$$\Pi'(m^2) = 0, \qquad \frac{d\Pi'}{dp^2}|_{p^2=m^2} = 0, \qquad \tilde{\Gamma}^{(4)}|_{s=4m^2} = -\lambda$$

(Continue next time.)