

1/31/11 Homework 3. Due Feb 7

1. As mentioned in lecture, the Coleman-Weinberg potential for  $V_{int} = \frac{\lambda}{4!}\phi^4$  is

$$\begin{aligned} V_1(\bar{\phi}) &= i \sum_{n=1}^{\infty} \frac{1}{2n} \int \frac{d^4 k}{(2\pi)^4} \left( \lambda \frac{1}{k^2 - m^2 + i\epsilon} \frac{\bar{\phi}^2}{2} \right)^n \\ &= \frac{1}{2} \int \frac{d^4 k_E}{(2\pi)^4} \ln \left( 1 + \frac{\frac{1}{2}\lambda\bar{\phi}^2}{k_E^2 + m^2} \right) \end{aligned}$$

(S. Coleman and E. Weinberg.) Symmetry factors:  $1/n!$  not all the way cancelled, because of  $Z_n$  rotation symmetry, and reflection, gives  $1/2n$ . At each vertex, can exchange external lines, so  $1/4!$  not all the way cancelled, leads to  $1/2$  for each vertex. Here you will derive the last expression above for  $V_1(\bar{\phi})$  another way. Write

$$e^{iW[J]} = \int [d\phi] e^{i(S + \int J\phi)/\hbar}$$

and expand  $\phi(x) = \bar{\phi} + \eta(x)$ , treating  $\bar{\phi}$  as a constant and imagining  $\eta$  to be a small fluctuation, and keeping only terms to order  $\eta^2$ . Do the Gaussian integral over  $\eta$  formally. Then use the relation given in class to convert  $W[J]$  into  $\Gamma[\bar{\phi}]$  to finally reproduce the above expression for  $V_1(\bar{\phi})$ . Hint: use  $\ln \det B = \text{Tr} \ln B$  for any operator  $B$ , so e.g.  $\ln \det(\partial^2 + m^2) = V_4 \int \frac{d^4 k}{(2\pi)^4} \ln(-k^2 + m^2)$  (where  $V_4$  is a spacetime box size which can be taken to infinity at the end of the day; it cancels anyway).

2. For a scalar field theory, with  $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$ , show that the EOM are satisfied, up to a contact term:

$$\frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x_\mu} \langle T\phi(x)\phi(y) \rangle + \langle V'(\phi(x))\phi(y) \rangle = \alpha\delta(x-y).$$

To do this problem, consider the functional integral for  $W[J]$ , and use the invariance of the functional integral under a change of variables. The change of variables is  $\phi \rightarrow \phi + \epsilon f(x)$ , where  $f(x)$  is an arbitrary function of  $x$ , and  $\epsilon$  is an infinitesimal parameter (drop terms of order  $\epsilon^2$  and higher). Derive in this way the above result (and determine the coefficient  $\alpha$ ). (Don't worry about the Jacobian in the  $[d\phi]$  integration measure - it doesn't contribute.) The source term for  $\phi$  is  $J$ , and you can think of  $f(x)$  as a source for the EOM.

- 3 (This problem is similar to problem 2.) Consider  $\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \frac{\lambda}{4}\phi^2\phi^{*2}$ , where  $\phi$  is a complex scalar field. The lagrangian has a symmetry,  $\phi \rightarrow e^{ia}\phi$ , so there

is a corresponding conserved current  $J^\mu(x)$ , with  $\partial_\mu J^\mu = 0$ . Find  $J^\mu(x)$ , using the Noether method. Now, in analogy with the above problem, use the functional integral to show that the current is conserved in correlation functions, up to contact terms:

$$\frac{\partial}{\partial x} \langle T J^\mu(x) \phi(y) \rangle = \beta \delta(x - y).$$

Derive this similarly to problem 2 (and, again determine the constant  $\beta$ ).