2/3/10 Lecture 9 outline

- Recall from last time: the 1-loop term in $\Gamma^{(2)}$ for $\lambda\phi^4$

$$\Pi'(p^2) = \tfrac{1}{2}\lambda \int \frac{d^4 k_E}{(2\pi)^4} \frac{1}{k_E^2 + m^2} + \text{more loops.}$$

Let's illustrate another, extremely popular, choice of regulator: dimensional regularization. Suppose that we had D instead of 4 dimensions. Compute by analytic continuation in D. Then take $D = 4 - \epsilon$, and take $\epsilon \to 0$. By going slightly below 4 dimensions, we improve the UV behavior (make the theory weaker in the UV, though stronger in the IR).

So we write

$$I \equiv \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{k_E^2 + m^2} = \frac{\Omega_{D-1}}{(2\pi)^D} \int_0^\infty u^{D-1} du \frac{1}{u^2 + m^2}.$$

Again, $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit sphere S^{D-1}. Let $u^2 = m^2 y$

$$I = \frac{m^{D-2}}{2^D \pi^{D/2} \Gamma(D/2)} \int_0^\infty \frac{y^{(D-2)/2} dy}{y+1}.$$

Now use $(y-1)^{-1} = \int_0^\infty dt e^{-t(y-1)}$ and $\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1}$ to get

$$I = \frac{m^{D-2}}{(4\pi)^{D/2}} \Gamma(1 - \tfrac{1}{2}D).$$

This blows up for $D = 4$, because $\Gamma(1 - \frac{1}{2}D)$ has a pole there. Recall $\Gamma(z)$ has a simple pole at $z = 0$, and also at all negative integer values of z.

Recall that near $x = 0$,

$$\lim_{x \to 0} \Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x),$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. For $x = -n$, we can write a similar expression, which also follows from the above and $\Gamma(z+1) = z\Gamma(z)$. This gives

$$\lim_{x \to -n} \Gamma(x) = \frac{(-1)^n}{n!} (\frac{1}{x+n} - \gamma + 1 + \ldots + \frac{1}{n} + \mathcal{O}(x+n).).$$

E.g. use $\Gamma(2 - D/2) = (1 - D/2)\Gamma(1 - D/2)$. Let $D = 4 - \epsilon$, then (dropping $\mathcal{O}(\epsilon)$,

$$\frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}} \Delta^{D/2-2} \to \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \log \frac{\Delta}{4\pi} - \gamma \right).$$

We can apply this to evaluate $\Pi^{(1)}(p^2)$. One last thing: replace $\lambda_{old} = \lambda_{new}\mu^{4-D}$, where λ_{new} is dimensionless. Expanding around $D = 4$, we get

$$\Pi'(p^2)^{(1)} = -\frac{\lambda m^2}{32\pi^2}\left(\frac{2}{\epsilon} - \log\frac{m^2}{4\pi\mu^2} + 1 - \gamma\right).$$

- More useful integrals:

$$\int \frac{d^D k_E}{(2\pi)^D}\frac{1}{(k_E^2+\Delta)^n} = \frac{1}{4\pi)^{D/2}}\frac{\Gamma(n-\frac{1}{2}D)}{\Gamma(n)}\Delta^{D/2-n}.$$

$$\int \frac{d^D k_E}{(2\pi)^D}\frac{k_E^2}{(k_E^2+\Delta)^n} = \frac{1}{4\pi)^{D/2}}\frac{D}{2}\frac{\Gamma(n-\frac{1}{2}D-1)}{\Gamma(n)}\Delta^{1+D/2-n}.$$

- Now consider $\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)$. There are three 1-loop diagrams, in the s, t, u channels. Recall $s = (p_1+p_2)^2$, $t = (p_1+p_3)^2$, $u = (p_1+p_4)^2$, $s+t+u = 4m^2$. Get

$$\tilde{\Gamma}^{(4)} = -\lambda\hbar^{-1} + (-i\lambda)^2(F(s)+F(t)+F(u)) + O(\hbar),$$

where

$$F(p^2) = \tfrac{1}{2}i\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}\frac{1}{(k+p)^2-m^2}.$$

The $\frac{1}{2}$ is a symmetry factor. Evaluate using

$$\frac{1}{AB} = \int_0^1 dx\frac{1}{(xA+(1-x)B)^2}.$$

Aside: more generally, have

$$\prod_{j=1}^n A_j^{-\alpha_j} = \frac{\Gamma(\sum_j\alpha_j)}{\prod_j\Gamma(\alpha_j)}\int_0^1 dx_1\ldots\int_0^1 dx_n\delta(1-\sum_j x_j)\frac{\prod_k x^{\alpha_k-1}}{(\sum_i x_iA_i)^{\sum\alpha_j}}.$$

Get

$$F(p_E^2) = -\int\frac{d^4k_E}{(2\pi)^4}\int_0^1 dx\frac{1}{(xk_E^2+m^2+(1-x)(k_E+p_E)^2)^2}.$$

The quantity in the denominator is $k_E^2+(1-x)2k_E\cdot p_E+(1-x)p_E^2+m^2 = (k_E+(1-x)p_E)^2+p_E^2(1-x)x+m^2$, so

$$F(s_E) = -\int\frac{d^4k_E}{(2\pi)^4}\int_0^1 dx\frac{1}{(k_E^2+m^2+x(1-x)s_E)^2}.$$

Where $s_E = p_E^2 = -s$. Evaluate the k integral using the dimreg integrals above. Expanding around $D = 4 - \epsilon$, this gives

$$F(s_E) = -\frac{1}{32\pi^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log(m^2 + x(1-x)s_E) \right).$$

So the one-loop contribution to $\tilde{\Gamma}^{(4)}$ is

$$\frac{\lambda^2}{32\pi^2} \left(\frac{2}{\epsilon} - \gamma + \log \frac{4\pi\mu^2}{m^2} - \int_0^1 dx \log(1 + x(1-x)\frac{s_E}{m^2}) \right) + (s \to t) + (s \to u).$$

The integral is evaluated using

$$\int_0^1 dx \log(1 + \frac{4}{a} x(1-x)) = -2 + \sqrt{1+a} \log \left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1} \right) \qquad a > 0.$$