1/8/10 Lecture 2 outline

⋆ Reading: Srednicki ch. 7.

• Last time: the path integral automatically gives the time ordered products that we'll need for computing S-matrix elements. This time: introduce **sources** for the fields as a trick to get the time order products from derivatives of a generating function (or functional).

• Consider QM with Hamiltonian $H(q,p)$, modified by introducing a source for q, $H \to H + f(t)q$. (We could also add a source for p, but don't bother doing so here. See Srednicki.). Consider moreover replacing $H \to H(1-i\epsilon)$, with $\epsilon \to 0^+$, which has the effect of projecting on to the ground state at $t \to \pm\infty$. As mentioned last lecture, this'll be related to the $i\epsilon$ of the Feynman propagator. Consider the vacuum-to vacuum amplitude in the presence of the source,

$$\langle 0|0\rangle_f = \int [dq] \exp[i \int dt(L + f(t)q)/\hbar] \equiv Z[f(t)].$$

Once we compute $Z[f(t)]$ we can use it to compute arbitrary time-ordered expectation values. Indeed, $Z[f]$ is a generating functional[1] for time ordered expectation values of products of the $q(t)$ operators:

$$\langle 0| \prod_{j=1}^{n} T q(t_j)|0\rangle = \prod_{j=1}^{n} \frac{1}{i}\frac{\delta}{\delta f(t_j)} Z[f]\big|_{f=0},$$

where the time evolution $e^{-iHt/\hbar}$ is accounted for on the LHS by taking the operators in the Heisnberg picture.

We'll be interested in such generating functionals, and their generalization to quantum field theory (replacing $t \to (t, \vec{x})$).

• We can explicitly evaluate the generating functional for the case of gaussian integrals, e.g. the harmonic oscillator example mentioned above.

To see how, let's first consider ordinary (non functional), multi-dimensional gaussian integrals:

$$\prod_{i=1}^{N} d\phi_i \exp(-(\phi, B\phi)) = \pi^{N/2}(\det B)^{-1/2},$$

where $(\phi, B\phi) = \sum_i \phi_i (B\phi)_i$ and $(B\phi)_i = \sum_j B_{ij}\phi_j$. The integral was evaluated by changing variables in the $d\phi_i$, to the eigenvectors of the symmetric matrix B; then the integrals

[1] Recall how functional derivatives work, e.g. $\frac{\delta}{\delta f(t)} f(t') = \delta(t-t')$.

decouple into a product of simple 1-variable gaussians. As before, we'll be interested in the gaussians with an i in the exponent, which we evaluate as mentioned before,

$$\int_{-\infty}^{\infty} d\phi \exp(ia\phi^2) = \sqrt{\frac{i\pi}{a}}.$$

where we analytically continued from the case of an ordinary gaussian integral. Think of a as being complex. Then the integral converges for $\mathrm{Im}(a) > 0$, since then it's damped. To justify the above, for real a, we need the integral to be slightly damped, not just purely oscillating. To get this, take $a \to a + i\epsilon$, with $\epsilon > 0$, and then take $\epsilon \to 0^+$.

• Now discuss generating functions. First consider ordinary (non-functional) gaussian integrals. We'd like to evaluate integrals like

$$\prod_{i=1}^{N} \int d\phi_i f(\phi_i) \exp(-(\phi, B\phi))$$

for functions, like products of the ϕ_i. We can do this by computing a generating function:

$$\prod_{i=1}^{N} \int d\phi_i f(\phi_i) \exp(-B_{ij}\phi_i\phi_j) = f(\frac{\partial}{\partial J_i})Z(J_i)\big|_{J_i=0}$$

Where

$$Z(J_i) \equiv \prod_{i=1}^{N} \int d\phi_i \exp(-B_{ij}\phi_i\phi_i + J_i\phi_i)$$

Evaluate via completing the square: the exponent is $-(\phi, B\phi) + (J, \phi) = -(\phi', B\phi') + \frac{1}{4}(J, B^{-1}J)$, where $\phi' = \phi - \frac{1}{2}B^{-1}J$. So

$$Z(J_i) = \prod_{i=1}^{N} \int d\phi_i \exp(-B_{ij}\phi_i\phi_j + J_i\phi_i) = \pi^{N/2}(\det B)^{-1/2} \exp(B^{-1}_{ij} J_i J_j/4)$$

We'll want to compute amplitudes like

$$\frac{\langle 0|\prod_i Tq(t_i)|0\rangle_{J=0}}{\langle 0|0\rangle_{J=0}}$$

and for these the $\det B$ factor above will cancel between the numerator and the denominator. This is related to the cancellation of vacuum bubble diagrams. The important piece above is the exponent with the sources.

• Now let's apply the above to compute the generating functional for the example of QM harmonic oscillator (scaling $m = 1$),

$$Z[f(t)] = \int [dq(t)] \exp(-\frac{i}{\hbar}\int dt \left[\tfrac{1}{2}q(t)(\frac{d^2}{dt^2} + \omega^2)q(t) - f(t)q(t)\right]).$$

This is analogous to the multi-dimenensional gaussian above, where i is replaced with the continuous label t, $\sum_i \to \int dt$ etc. and the matrix B_{ij} is replaced with the differential operator $B \to \frac{i}{2\hbar}(\frac{d^2}{dt^2} + \omega^2)$. So doing the gaussian gives a factor of $\sqrt{\det B}$ which we don't need to compute now because it'll cancel, and the exponent with the sources from completing the square, which is the term we want. That involves B^{-1}, which we can compute by Fourier transforming. In the end, we get

$$\langle 0|0\rangle_f = \exp[\frac{i}{2}\int dtdt' f(t)G(t-t')f(t')],$$

with $G(t)$ the Green's function for the oscillator, $(\partial_t^2 + \omega^2)G(t) = \delta(t)$,

$$G(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iEt/\hbar}}{-E^2 + \omega^2 - i\epsilon} = \frac{i}{2\omega} e^{-i\omega|t|}. \tag{1}$$

• Now that we know the generating functional, we can use it to compute time ordered expectation values via

$$\langle 0|T\prod_{i=1}^{n} \phi_H(x_i)|0\rangle / \langle 0|0\rangle = Z_0^{-1}\int [d\phi]\prod_{i=1}^{n} \phi(x_i)\exp(iS/\hbar),$$

with $Z_0 = \int [d\phi]\exp(iS/\hbar)$.