3/5/10 Lecture 18 outline

• Last time: The photon has 1PI propagator $i\Pi^{\mu\nu}(k) = (p^2 g^{\mu\nu} - p^\mu p^\nu)\Pi(k^2)$. Summing these gives the full propagator. Writing it in Feynman gauge, get for the full propagator $-ig_{\mu\nu}/p^2(1-\Pi(p^2))$. Assuming that $\Pi(p^2)$ is regular at $p^2=0$, get pole at $p^2=0$ with residue $Z_3 \equiv (1-\Pi(0))^{-1}$.

Result to one loop from virtual electron/positron loop:

$$i\Pi^{\mu\nu}(q) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} tr\left(\gamma^\mu \frac{i}{\not{k}-m}\gamma^\nu \frac{i}{\not{k}+\not{q}-m}\right).$$

Combine denominators using Feynman parameter

$$\frac{1}{(k^2-m^2)((k+q)^2-m^2} = \int_0^1 dx \frac{1}{(\ell^2+x(1-x)q^2-m^2)^2}$$

with $\ell = k+xq$. Go to Euclidean space and do integrals using our previous tables of integrals in dim-reg to find

$$\Pi(p^2) = -\frac{8e^2}{(4\pi)^{d/2}}\Gamma(2-\tfrac{1}{2}d)\int_0^1 dx x(1-x)\Delta^{\frac{1}{2}d-2},$$

with $\Delta = m^2 - x(1-x)p^2$. Evaluating for $d=4-\epsilon$,

$$\Pi(p^2) = -\frac{2\alpha}{\pi}\int_0^1 dx x(1-x)\left(\frac{2}{\epsilon}-\gamma+\log(4\pi/\Delta)\right).$$

We now renormalize, $\psi_B = Z_2^{1/2}\psi_R$, $A_B^\mu = Z_3^{1/2}A_R^\mu$, $e_B Z_2 Z_3^{1/2} = e_R Z_1$. $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$.

In particular, the counter-term contributes to $i\Pi^{\mu^\nu}$ as $\delta\Pi = -(Z_3-1)$. So, to one loop, we get

$$\Pi(p^2) = -\frac{\alpha}{\pi}\epsilon^{-1}\frac{2}{3} + (Z_3-1)^{(1)} + \text{finite}.$$

in MS, choose Z_3 to cancel the $1/\epsilon$ term only, so $Z_3 - 1 = -\frac{\alpha}{\pi}\epsilon^{-1}\frac{2}{3}$.

We'll soon note that $e_{phys} = \sqrt{Z_3}e_B$, or better $\alpha = e_{phys}^2/4\pi = Z_3\mu^{-\epsilon}\alpha_B$. Write this as $\alpha_B = \alpha\mu^\epsilon Z_\alpha$, where

$$Z_\alpha \equiv Z_3^{-1} \equiv 1 + \sum_k a_k(\alpha)\epsilon^{-k}.$$

In particular, we found above that $a_1 = 2\alpha/3\pi$ to one-loop order.

Now it's just like what we did in $\lambda\phi^4$, use the fact that α_B is independent of μ to get

$$0 = \epsilon\alpha Z_3^{-1} + \beta(\alpha,\epsilon)Z_3^{-1} + \beta(\alpha,\epsilon)\alpha\frac{d}{d\alpha}Z_3^{-1}.$$

where $\beta(\alpha, \epsilon) = d\alpha/d\ln\mu$. To have a smooth $\epsilon \to 0$ limit, we need

$$\beta(\alpha, \epsilon) = -\epsilon\alpha + \beta(\alpha),$$

$$\beta(\alpha) = \alpha^2 \frac{da_1}{d\alpha}.$$

Using the above result for a_1, we get finally

$$\beta(\alpha) = \frac{d\alpha}{d\ln\mu} = \frac{2\alpha^2}{3\pi} + \text{higher loops}.$$

This is the promised beta function of QED. It's positive, as in $\lambda\phi^4$, and every other theory except non-Abelian gauge theories. Its sign is again related to charge screening, so the effective charge is small at long distances (IR free) and blows up at short distances (the Landau pole), as we discussed before. Integrate 1-loop beta function:

$$\alpha^{-1}(\mu) = -\frac{2}{3\pi}\ln(\frac{\mu}{\Lambda}).$$

Makes sense only for $\mu < \Lambda$, i.e. in the IR. Λ is a UV cutoff. Get $\alpha \to \infty$ as $\mu \to \Lambda$; this is the Landau pole. Looks bad, but we'll see the the energy scale where it blows up is so fantastically large that we don't need to worry (something new should fix it in the UV, e.g. grand unification can do the job). It does not run to zero in the IR, because there are no massless charged particles. It runs toward zero until it gets to the energy scale of the lightest charged particle, $m_e = 0.5MeV$, and then it stops running. So $137 = \frac{3}{3\pi}\ln(\Lambda/m_e)$. Gives $\Lambda = m_e \exp(137\pi)$, which too huge to worry about the apparent Landau pole there. (Other charged particles will bring the scale of Λ down to $\Lambda = m_e \exp(137\pi/N_f)$ where N_f is the effective number of charged particles, but it's still huge.)

• Let's note some other interesting things about the finite part of $\Pi(p^2)$. $\Pi(p^2)$ has a branch cut starting at $p^2 = 4m^2$, and its imaginary part above and below the cut have

$$Im(\Pi(p^2 \pm i\epsilon) = \mp\frac{\alpha}{3}\sqrt{1-\frac{4m^2}{p^2}}(1+\frac{2m^2}{p^2}),$$

which is related by the optical theorem to the total cross section for creating an on-shell fermion-antifermion pair,

$$\frac{d\sigma}{d\Omega} = \frac{|\vec{p}|}{32\pi^2 s^{3/2}}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}|^2.$$

• Continue with QED renormalization. $\psi_B = Z_2^{1/2}\psi_R$, $A_B^\mu = Z_3^{1/2}A_R^\mu$, $e_B Z_2 Z_3^{1/2} = e_R Z_1$. $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$. We discussed Z_3 above, from the full photon propagator. Now consider the full electron propagator,

$$S(p) = \frac{i}{\not{p} - m - \Sigma(p) + i\epsilon},$$

where $-i\Sigma$ is the 1PI contribution to the propagator. E.g. to 1 loop get

$$-i\Sigma(p^2) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\mu\nu}}{k^2} \gamma^\mu \frac{i}{\not{p} - \not{k} - m} \gamma^\nu.$$

The function $S(p)$ has a pole at the physical mass, $m_{phys} = m + \Sigma(0)$, so the constant part of Σ shifts the mass. The $\sim \not{p}$ part of Σ renormalizes the residue of $S(p)$. The residue is iZ_2. Again, we can add counterterms to shift these and preserve a renormalization condition.

• 1PI vertex for electron interacting with photon, $-ie\Gamma^\mu(p', p)$. The tree-level term is $-ie\gamma^\mu$. The photon has momentum $q = p' - p$. Can show that Lorentz and kinematic structure is such that

$$Z_2 \Gamma^\mu(p', p) = \gamma^\mu F_1(q^2) + i\frac{\sigma^{\mu\nu} q_\nu}{2m} F_2(q^2),$$

where $\sigma^{\mu\nu} = \frac{1}{2}i[\gamma^\mu, \gamma^\nu]$ and F_i are "form factors." The electron has magnetic moment $\vec{\mu} = g(e\vec{S}/2m)$, with $g = 2 + 2F_2(0)$. The diagram for $F_2(0)$ at one-loop is convergent, and yields $F_2(0) = \alpha/2\pi$. The diagram for $F_1(q^2)$ is UV, and also IR divergent at $q^2 = 0$. Define $\Gamma^\mu(q^2 = 0) = Z_1^{-1}\gamma^\mu$.

• The Ward identity gives

$$S(p_k)[-iek_\mu \Gamma^\mu(p_k, p)]S(p) = e(S(p) - S(p_k)).$$

So

$$-ik_\mu \Gamma^\mu(p_k, p) = S^{-1}(p_k) - S^{-1}(p)$$

This gives $Z_1 = Z_2$. Thus $F_1(0) = 1$.

• Bare and renormalized fields, and counterterms. $\psi_B = Z_2^{1/2}\psi_R$, $A_B^\mu = Z_3^{1/2}A_R^\mu$, $e_B Z_2 Z_3^{1/2} = e_R Z_1$. $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$.

$$\mathcal{L}_R = -\frac{1}{4} F_{R\mu\nu} F_R^{\mu\nu} + \bar{\psi}_R(i\not{\partial} - e_R \not{A}_R - m_R)\psi_R,$$

$$\mathcal{L}_{ct} = -\frac{1}{4}\delta_3(F_{R\mu\nu})^2 + \bar{\psi}_R(i\delta_2\partial\!\!\!/ - \delta_1 e_R A\!\!\!/_R - \delta_m)\psi_R.$$

Where $\delta_1 = Z_1 - 1$, $\delta_2 = Z_2 - 1$, $\delta_3 = Z_3 - 1$, and $\delta_m = Z_2 m_0 - m$. We have

$$e_B Z_2 Z_3^{1/2} = e_R Z_1,$$

where the Z_1 will cancel the Z_1^{-1} in $\Gamma^\mu(q^2 = 0) = Z_1^{-1}\gamma^\mu$.

• Gauge invariance requires $Z_1 = Z_2$, since then $\delta_1 = \delta_2$ and the counterterm pieces have the same gauge invariance. Sure enough, direct calculation shows $Z_1 = Z_2$ (to all orders in perturbation, theory, and exactly)! This is a special case of a Ward identity, stating $\Gamma_\mu(p,p) = -\partial_{p^\mu}\Sigma(p)$. So get $e_{phys} = \sqrt{Z_3}e_B$, as promised.

So $e_R = \sqrt{Z_3}e_0 = e_{phys}$. Shows that renormalized charge is same for all species (e.g. electron and muon and anti-proton all have exactly the same effective charge).