2/22/10 Lecture 14 outline

- Last time:

$$\left(\frac{\partial}{\partial \ln\mu} + \beta(\lambda_R)\frac{\partial}{\partial\lambda_R} + \gamma_m m_R \frac{\partial}{\partial \ln m_R} - n\gamma\right)\tilde{\Gamma}_R^{(n)}(p_1,\dots p_n;\lambda_R, m_R,\mu) = 0$$

Here

$$\beta(\lambda) \equiv \frac{d}{d\ln\mu}\lambda_R$$

$$\gamma = \tfrac{1}{2}\frac{d}{d\ln\mu}\ln Z_\phi$$

$$\gamma_m = \frac{d\ln m_R}{d\ln\mu}.$$

- Understand what β and γ mean: the bare quantities are some function of the renormalized ones and epsilon. E.g. for $\lambda\phi^4$ in MS we have

$$\lambda_B = \mu^\epsilon(\lambda + \delta_\lambda) \equiv \mu^\epsilon \lambda Z_\lambda$$

Let us write

$$Z_\lambda \equiv 1 + \sum_k a_k(\lambda)\epsilon^{-k},$$

where we found $a_1(\lambda) = +3\lambda/16\pi^2$ to one loop. The bare parameter λ_B is independent of μ, whereas λ depends on μ, such that the above relation holds. Take $d/d\ln\mu$ of both sides,

$$0 = \epsilon\lambda Z_\lambda + \beta(\lambda,\epsilon)Z_\lambda + \beta(\lambda,\epsilon)\lambda\frac{dZ_\lambda}{d\lambda}.$$

Using the above expansion for Z_λ and requiring that $\beta(\lambda,\epsilon)$ be regular at $\epsilon = 0$, so $\beta(\lambda,\epsilon) = \beta(\lambda) + \sum_n \beta_n\epsilon^n$, gives

$$\beta(\lambda,\epsilon) = -\epsilon\lambda + \beta(\lambda)$$

$$\beta(\lambda) = \lambda^2\frac{da_1}{d\lambda}$$

$$\lambda^2\frac{da_{k+1}}{d\lambda} = \beta(\lambda)\frac{d}{d\lambda}(\lambda a_k).$$

The beta function is determined entirely from a_1. The $a_{k>1}$ are also entirely determined by a_1. In k-th order in perturbation theory, the leading pole goes like $1/\epsilon^k$.

We find for $\lambda\phi^4$

$$\beta(\lambda) = \frac{3\lambda^2}{16\pi^2} + \mathcal{O}(\lambda^3).$$

Integrating, this gives

$$\lambda = \lambda_0 \left(1 - \frac{3}{16\pi^3}\lambda_0 \ln(\mu/\mu_0)\right)^{-1}.$$

• We similarly have

$$\gamma_\phi(\lambda,\epsilon) = \tfrac{1}{2}\frac{d}{d\ln\mu}\ln Z_\phi$$

where

$$Z_\phi = 1 + \sum_k Z_\phi^{-k}(\lambda)\epsilon^{-k}.$$

So

$$\gamma_\phi(\lambda,\epsilon) = \tfrac{1}{2}\beta(\lambda,\epsilon)\frac{d}{d\lambda}\ln Z_\phi.$$

Using $\beta(\lambda,\epsilon) = -\epsilon\lambda + \beta(\lambda)$, we get

$$\gamma_\phi = -\tfrac{1}{2}\lambda\frac{d}{d\lambda}Z_\phi^{(1)}.$$

We similarly have $m_B^2 = (m^2 + \delta_{m^2})Z_\phi^{-1} \equiv Z_m m^2$ and

$$\gamma_m(\lambda) = \tfrac{1}{2}\frac{d\ln m^2}{d\ln\mu} = -\tfrac{1}{2}\frac{d\ln Z_m}{d\ln\mu} = -\tfrac{1}{2}\beta\frac{d\ln Z_m}{d\lambda} = \tfrac{1}{2}\lambda\frac{dZ_m^{(1)}}{d\lambda}$$

where $Z_m^{(1)}$ means the coefficient of $1/\epsilon$. In all these cases, only the coefficient of $1/\epsilon$ matters.

In particular, for $\lambda\phi^4$ we have

$$\gamma_m(\lambda) = \tfrac{1}{2}\lambda\frac{dZ_m^{(1)}}{d\lambda} = \tfrac{1}{2}\frac{\lambda}{16\pi^2} - \frac{5}{12}\frac{\lambda^2}{6(16\pi^2)^2} + \dots$$

where $Z_m^{(1)}$ means the coefficient of $1/\epsilon$ and ... are higher orders in perturbation theory, and

$$\gamma_\phi = -\tfrac{1}{2}\lambda\frac{d}{d\lambda}Z_\phi^{(1)} = \frac{1}{12}\frac{\lambda^2}{(16\pi^2)^2} + \dots$$

For any gauge invariant field ϕ, we always have $\gamma_\phi \geq 0$, where $\gamma_\phi = 0$ iff it is a free field. This follows from the spectral decomposition result that $Z \leq 1$.

• Note: $\beta > 0$ means the coupling is small in the IR, and large in the UV. Such theories are “not asymptotically free” or are “IR free.” Most theories are like this, e.g. $\lambda\phi^4$, QED, Yukawa interactions. If $\beta < 0$, then the coupling is small in the UV, and large in the IR. Such theories are “asymptotically free;” only non-Abelian gauge theories, like QCD, are like that.