215b Homework 1, due Jan. 13

1. In class, I sketched how to compute $U(x_b, t_b; x_a, t_a) = \langle x_b | e^{-iH(t_b - t_a)/\hbar} | x_a \rangle$ using the path integral, $U = \int [dx] \exp(iS/\hbar)$ for a free-particle. The result was given in class.

(a) As mentioned, the result for this case of a free-particle happens to be of the form $U = F(t_b - t_a) e^{iS_{cl}/\hbar}$. Verify this, by computing S_{cl} for the free particle. Verify also that $p = \partial S_{cl}/\partial x_b$ and $E = -\partial S_{cl}/\partial t_b$, as stated in class.

(b) Re-derive the expression given in class for $U = \langle x_b | e^{-i\widehat{H}(t_b - t_a)/\hbar} | x_a \rangle$, for the case of a free-particle, by using the standard operator description of QM with $\widehat{H} = \widehat{p}^2/2m$. Do this by introducing a complete set of momentum eigenstates. In this way, you'll also verify that the normalization factor given in class for the integral is indeed correct.

2. (taken from Peskin and Schroeder, problem 9.2)

(a) Evaluate the quantum statistical partition function

$$Z = \mathrm{Tr} e^{-\beta H}$$

using the strategy which led to the path integral (introducing lots of complete sets of states) for evaluating the matrix elements of $e^{-iHt/\hbar}$ in terms of functional integrals. Show that one again finds a functional integral, over functions defined on a domain that is of length β and periodically connected in the time direction. Note that the Euclidean form of the Lagrangian appears in the weight.

(b) Evaluate this integral for the simple harmonic oscillator, $L_E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2$ by introducing a Fourier decomposition of $x(t)$:

$$x(t) = \sum_n x_n \frac{1}{\sqrt{\beta}} e^{2\pi i n t/\beta}.$$

The dependence of the result on β is a bit subtle to obtain explicitly, since the measure for the integral over $x(t)$ depends on β in any discretization. However, the dependence on ω should be unambiguous. Show that up to a (possibly divergent and β dependent) constant the integral reproduces exactly the familiar expression for the quantum partition function of an oscillator. [You may find the identity

$$\sinh z = z \cdot \prod_{n=1}^{\infty} \left(1 + \frac{x^2}{(n\pi)^2}\right)$$

useful.]

In class we discussed how

$$\langle 0|0\rangle_f = \int [dq] \exp[i \int dt (L + f(t)q)/\hbar] \equiv Z[f(t)].$$

$Z[f]$ is a generating functional for time ordered expectation values of products of the $q(t)$ operators:

$$\langle 0| \prod_{j=1}^{n} Tq(t_j)|0\rangle = \prod_{j=1}^{n} \frac{1}{i}\frac{\delta}{\delta f(t_j)} Z[f]\big|_{f=0}, \tag{1}$$

where the time evolution $e^{-iHt/\hbar}$ is accounted for on the LHS by taking the operators in the Heisnberg picture. We consider the harmonic oscillator in quantum mechanics ("SHO"), and motivated the result

$$Z_{SHO}[f] = \langle 0|0\rangle_f = \exp[\frac{i}{2}\int_{-\infty}^{\infty} dt dt' f(t) G(t-t') f(t')], \tag{2}$$

with (setting $\hbar = 1$)

$$G_{SHO}(t) = \int_{-\infty}^{\infty} \frac{dE}{2\pi} \frac{e^{-iEt}}{-E^2 + \omega^2 - i\epsilon} = \frac{i}{2\omega} e^{-i\omega|t|}. \tag{3}$$

Verify explicitly that, using (1) and (2) it follows that e.g.

$$\langle 0|Tq(t_1)q(t_2)|0\rangle = -iG_{SHO}(t_2 - t_1), \tag{4}$$

and

$$\langle 0|Tq(t_1)q(t_2)q(t_3)q(t_4)|0\rangle = (-i)^2 \left(G_{SHO}(t_2-t_1)G_{SHO}(t_3-t_4) + \text{perms}\right) \tag{5}$$

where perms means two similar terms, with $t_2 \leftrightarrow t_3$ and $t_2 \leftrightarrow t_4$.

3. This problem is taken from (Srednicki, problem 7.3).

(a) Use the Heisenberg equations of motion, $\dot{A} = i[H, A]$ to find explicit expressions for $\dot{q}$ and p for the harmonic oscillator. Solve to get the Heisenberg picture operators $q(t)$ and $p(t)$ in terms of the Schrodinger picture operators q and p.

(b) Using the result of part (a), write the Heisenberg picture operators in terms of the usual SHO creation and annihilation operators a and $a^\dagger$.

(c) Use the results from the above parts, and $a|0\rangle = 0$, to verify (4) and (5).