1/22/09 Lecture 7 outline

• We normalize $Z[J=0]=1$, since we anyway divide by the vacuum-to-vacuum amplitude. This recovers the story of cancellation of bubble diagrams. For computing S-matrix elements, we will especially be interested in *connected* Green's functions. There are nice combinatoric formulae (you might have already seen some last quarter?). E.g.

$$\sum \text{all diagrams} = \Big(\sum \text{``connected''}\Big) \cdot \exp(\sum \text{disconnected vacuum bubbles}).$$

And the vacuum bubble diagrams cancel. We write "connected" because for $n > 2$ point functions there are still disconnected diagrams, connected to the external points, included in this sum. But even those disconnected diagrams drop out when we consider $S-1$: they correspond to the 1. In the end, we're interested in the fully connected diagrams. There is a generating functional for them. (N.B. sometimes people reverse the names of what I'm calling W and Z!. Peskin calls $W \to E$.) Defining,

$$iW[J] \equiv \ln Z[J]$$

$iW[J]$ is the generating functional for connected Green's functions

$$G^{(n)}_{conn}(x_1, \dots x_n) = \hbar^{-1} \prod_{j=1}^{n} \frac{-i\delta}{\delta J(x_j)} iW[J],$$

i.e.

$$iW[J] = \hbar \sum_{n=1}^{\infty} \frac{i^n}{n!} \int d^4x_1 \dots d^4x_n G^{(n)}_{conn}(x_1, \dots x_n) J(x_1) \dots J(x_n).$$

In momentum space, we can write:

$$iW[J] = \hbar \sum_{n=1}^{\infty} \frac{i^n}{n!} \int \frac{d^4k_1}{(2\pi)^4} \dots \frac{d^4k_n}{(2\pi)^4} \tilde{J}(-k_1) \dots \tilde{J}(-k_n) \tilde{G}_c(k_1, \dots k_n).$$

• Examples, to illustrate how $iW[J] \equiv \ln Z[J]$ gives the connected diagrams. First

$$-i\frac{\delta iW}{\delta J} = \frac{1}{Z[J]}\frac{\delta Z[J]}{\delta J(x)} = \langle \phi(x) \rangle_J.$$

$$(-i)^2 \frac{\delta^2}{\delta J(x)\delta J(y)}(iW) = \langle \phi(x)\phi(y)\rangle_J - \langle \phi(x)\rangle_J \langle \phi(y)\rangle_J.$$

Note that $\langle \phi(x)\phi(y)\rangle$ has two types of contributions, connected and disconnected; the 2nd term precisely cancels off the disconnected ones. Similarly $\delta W/\delta J^3$ has terms like

$\langle\phi\phi\phi\rangle - \langle\phi\phi\rangle\langle\phi\rangle + 2\langle\phi\rangle\langle\phi\rangle\langle\phi\rangle$, which give precisely $\langle\phi\phi\phi\rangle_{connected}$. Can prove by induction that the *log* in W properly subtracts away all non-connected diagrams!

• Will later discuss LSZ: how to relate Green's functions to S-matrix elements (and hence physical observables). Will see there that only connected diagrams contribute; this is why W is useful.

• Let's write

$$e^{iW[J]} = N\int [d\phi] e^{\frac{i}{\hbar}\left(S[\phi]+\int J\phi\right)},$$

(here we rescaled J by factor of $1/\hbar$ compared with before).

• Suppose diagram has I internal lines, V vertices, L loops. Connected graphs have $L = I - V + 1$. Graphs go like $\hbar^{-V}\hbar^{I} = \hbar^{L-1}$. So $W[J] = W_{-1}\hbar^{-1} + W_0 + \hbar W_1 + \ldots$, where W_{-1} are tree-graphs (no loops), W_0 gives the 1-loop graphs, etc.

• Example: free Klein Gordon theory. We found $Z[J]$ above. Then

$$W[J] = i\tfrac{1}{2}\hbar^{-1}\int d^4x \int d^4y J(X) D_F(x-y) J(y).$$

(Rescaled source J compared with before.)

We see that the only connected Green's function in this case is the 2-point function:

$$G^{(2)}_{free}(x,y) \equiv G(x-y) = \hbar D_F(x-y).$$

In an interacting theory, like $\lambda\phi^4$,

$$G^{(2)}(x,y) = \hbar D_F(x-y) + O(\lambda) \text{ corrections.}$$

• Emphasize that tree graphs are classical. Example: consider $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 + \phi J$, with the source term J. The classical field EOM is

$$(\partial_\mu\partial^\mu + m^2)\phi_c = -\frac{1}{3!}\lambda\phi_c^3 + J(x).$$

We can solve this classically to zero-th order in λ as

$$\phi_c^{(0)}(x) = \int d^4y D_F(x-y) iJ(y),$$

where $(\partial_\mu\partial^\mu + m^2)D_F(x-y) = -i\delta(x-y)$. To solve to next order in λ, we plug this back into the above:

$$\phi_c^{(1)}(x) = \phi_c^{(0)}(x) - i\frac{1}{3!}\lambda\int d^4y D_F(x-y)\phi_c^{(0)}(y)^3$$

Continue this way, this can be represented as a sum of tree-level diagrams, with one ϕ and different numbers of J's on the external legs. This is perturbation theory for the classical field theory.

• Examples of diagrams contributing to $G^{(n)}_{conn}$ for $n = 2, 4, 6$, in $\lambda\phi^4$.

• We have seen that the loop expansion is an expansion in powers of $\hbar$, since diagrams go like $\hbar^{L-1}$. Question: are we expanding in $\hbar$ (loops), or in powers of the small coupling constants, or both? Answer: it's generally the same expansion. Consider e.g. $\lambda\phi^r$ interaction. Then a connected diagram with E external lines (amputating their propagators) and I internal lines and V vertices is $\sim \hbar^{I-V}\lambda^V$. Now we use $L = I - V + 1$ and $E + 2I = rV$ (conservation of ends of the lines) to get that the diagram is $\sim \left(\hbar\lambda^{2/(r-2)}\right)^{L-1}\lambda^{E/(r-2)}$, so for fixed E the loop expansion is an expansion in powers of the effective coupling $\alpha \sim \hbar\lambda^{2/r-2}$.