1/13/09 Lecture 4 outline

• On to QFT and the Klein-Gordon theory,

$$Z_0 = \int [d\phi] e^{iS/\hbar} \qquad S = \tfrac{1}{2}\int d^4x \phi(x)(-\partial^2 - m^2)\phi(x),$$

where we integrated by parts and dropped a surface term.

Recall

$$\int_{-\infty}^{\infty} d\phi \exp(ia\phi^2) = \sqrt{\frac{i\pi}{a+i\epsilon}}.$$

where we analytically continued and for real a, we need the integral to be slightly damped, not just purely oscillating, so $a \to a + i\epsilon$, with $\epsilon > 0$, and then take $\epsilon \to 0^+$. Now we should do the same thing for our functional integral above. I.e. we should take $S = \frac{1}{2}\int d^4x\phi(x)(-\partial^2 - m^2 + i\epsilon)\phi(x)$, with $\epsilon > 0$, and then $\epsilon \to 0^+$. Note that the operator is $B \sim -\partial^2 - m^2 + i\epsilon$, which in momentum space is $p^2 - m^2 + i\epsilon$. Looks familiar! The Feynman $i\epsilon$ prescription, which you understood last quarter as needed to give correct causal structure of greens functions, here comes simply from ensuring that the integrals converge! This is why the path integral automatically gives the time ordering of the products. So

$$Z_0 = \mathrm{const}(\det(-\partial^2 - \mathrm{m}^2 + \mathrm{i}\epsilon))^{-1/2}.$$

• Now discuss generating functions, recalling the ordinary (non-functional) gaussian integrals:

$$Z(J_i) = \prod_{i=1}^{N} \int d\phi_i \exp(-B_{ij}\phi_i\phi_j + J_i\phi_i) = \pi^{N/2}(\det B)^{-1/2}\exp(B^{-1}_{ij}J_iJ_j/4)$$

Similarly, we can compute field theory Green's functions via the functional integral analog of the above. The generating functional is

$$Z[J(x)] = \int [d\phi] \exp(i\int d^4x[\mathcal{L} + J(x)\phi(x)]).$$

This is a functional: input function $J(x)$ and it outputs a number. Use it to compute

$$\langle 0|T\prod_{i=1}^{n}\phi(x_i)|0\rangle/\langle 0|0\rangle = Z[J]^{-1}\prod_{j=1}^{n}\left(-i\frac{\delta}{\delta J(x_i)}\right)Z[J]\big|_{J=0}.$$

E.g. free field theory, in analogy with the above, we have $B = (-i/2\hbar)(-\partial^2 - m^2 + i\epsilon)$, so $B^{-1} = 2i\hbar(-\partial^2 - m^2 + i\epsilon)^{-1}$. We then get for the generating functional

$$Z[J] = Z_0\exp(-\tfrac{1}{2}\hbar\int d^4xd^4yJ(x)D_F(x-y)J(y)),$$

with

$$D_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2-m^2+i\epsilon},$$

D_F gives $i(-\partial^2 - m^2 + i\epsilon)^{-1}$.

Can use this generating function to compute free field time ordered products.. it reproduces Wick's theorem.

Compute the Greens functions

$$G^{(n)}(x_i) \equiv \langle 0|T\prod_{i=1}^n \phi(x_i)|0\rangle/\langle 0|0\rangle$$

in free theory example. Find e.g. $G_0^{(2)}(x,y) = -i\hbar D_F(x-y)$, (where the subscript is to remind us it's the free theory), $G_0^{(4)} = G_0^{(2)}(x_1,x_2)G_0^{(2)}(x_3,x_4)+$ 2 permutations, etc.

More generally, including interactions, the path integral reproduces the results of using Dyson and Wick's theorem, or alternatively of Feynman diagrams.

To see this, notice that

$$\int [d\phi] \exp(\frac{i}{\hbar}[S_{free} + S_{int}[\phi] + \hbar \int d^4x J\phi]) = \exp[\frac{i}{\hbar}S_{int}[-i\frac{\delta}{\delta J}])Z_{free}[J].$$

So

$$Z[J] = N \exp[\frac{i}{\hbar}S_{int}[-i\frac{\delta}{\delta J}])Z_{free}[J],$$

where N is an irrelevant normalization factor (independent of J). Correspondingly, the green's functions are given by Correspondingly, the green's functions are given by

$$\begin{aligned} G^{(n)}(x_1 \dots x_n) &= \frac{\int [d\phi]\phi(x_1)\dots\phi(x_n)\exp(\frac{i}{\hbar}S_I[\phi])\exp[\frac{i}{\hbar}S_{free}]}{\int [d\phi]\exp(\frac{i}{\hbar}S_I[\phi])\exp[\frac{i}{\hbar}S_{free}]}, \\ &= \frac{1}{Z[J]}\prod_{j=1}^n \left(-i\hbar\frac{\delta}{\delta J(x_j)}\right)\cdot Z[J]\big|_{J=0}. \end{aligned}$$

(The denominator (in both lines) cancels off the vacuum bubble diagrams, which don't depend specifically on the Green's function.)