1/7/09 Lecture 2 outline

• Last time: For particles, consider the kernal of the time evolution operator

$$U(q_a, q_b; T) = \langle q_b | e^{-iHT/\hbar} | q_a \rangle.$$

Satisfies SE

$$i\hbar \partial_T U = HU.$$

Feynman:

$$U(q_a, q_b; T) = \int [dq(t)] e^{-S[x(t)]/\hbar}.$$

We showed how to derive this result from the operator description of QM by introducing lots of complete sets of position and momentum eigenstates, at infinitesimal time slices.

• Likewise, the same derivation leads to e.g.

$$\langle q_4, t_4 | T\widehat{q}(t_3)\widehat{q}(t_2) | q_1, t_1 \rangle = \int [dq(t)] q(t_3) q(t_2) e^{iS/\hbar},$$

where the integral is over all paths, with endpoints at (q_1, t_1) and (q_4, t_4).

Note that the LHS involves time ordered operators, while the RHS has a functional integral, which does not involve operators (so there is no time ordering). The fact that the time ordering comes out on the LHS is good, since we already reviewed last time that we'll need that for using the LSZ formula to compute quantum field theory amplitudes.

• Computation. Integral can be broken into time slices, as way to define it. E.g. free particle

$$\left(\frac{-im}{2\pi\hbar\epsilon}\right)^{N/2} \int \prod_{i=1}^{N-1} dx_i \exp[\frac{im}{2\hbar\epsilon} \sum_{i=1}^{N} (x_i - x_{i-1})^2]$$

Where we take $\epsilon \to 0$ and $N \to \infty$, with $N\epsilon = T$ held fixed.

Do integral in steps. Apply expression for real gaussian integral (valid: analytic continuation):

$$\int_{-\infty}^{\infty} d\phi \exp(ia\phi^2) = \sqrt{\frac{i\pi}{a}}.$$

where we analytically continued from the case of an ordinary gaussian integral. Think of a as being complex. Then the integral converges for $\mathrm{Im}(a) > 0$, since then it's damped. To justify the above, for real a, we need the integral to be slightly damped, not just purely oscillating. To get this, take $a \to a + i\epsilon$, with $\epsilon > 0$, and then take $\epsilon \to 0^+$.

After $n-1$ steps, get integral:

$$\left(\frac{2\pi i\hbar n\epsilon}{m}\right)^{-1/2}\exp[\frac{m}{2\pi i\hbar n\epsilon}(x_n-x_0)^2].$$

So the final answer is

$$U(x_b,x_a;T)=\left(\frac{2\pi i\hbar T}{m}\right)^{-1/2}\exp[im(x_b-x_a)^2/2\hbar T].$$

Note that the exponent is $e^{iS_{cl}/\hbar}$, where S_{cl} is the classical action for the classical path with these boundary conditions. (More generally, get a similar factor of $e^{iS_{cl}/\hbar}$ for interacting theories, from evaluating path integral using stationary phase.)

Plot phase of Uas a function of $x=x_b-x_a$, fixed T, Lots of oscillates. For large x, nearly constant wavelength λ, with

$$2\pi=\frac{m(x+\lambda)^2}{2\hbar T}-\frac{2m^2}{2\hbar T}\approx\frac{mx\lambda}{\hbar T}=p\lambda/\hbar.$$

Gives $p=\hbar k$.

Recover $\psi\sim e^{ipx/\hbar}$. More generally, get $p=\hbar^{-1}k$, with $p=\partial S_{cl}/\partial x_b$ (can show $p=\partial L/\partial\dot{x}=\partial S_{cl}/\partial x_b$. Can also recover $\psi\sim e^{-i\omega T}$, with $\omega=\hbar^{-1}(-\partial S_{cl}/\partial t_b)$. Agrees with $E=\hbar\omega$, since $E=p\dot{x}-L=-\partial S_{cl}/\partial t_b$.

• Nice application: Aharonov-Bohm. Recall $L=\frac{1}{2}m\dot{\vec{x}}^2+q\dot{\vec{x}}\cdot\vec{A}-q\phi$. Solenoid with $B\neq 0$ inside, and $B=0$ outside. Phase difference in wavefunctions is

$$e^{i\Delta S/\hbar}=e^{iq\oint\vec{A}\cot d\vec{x}/\hbar}=e^{iq\Phi/\hbar}.$$

Aside on Dirac quantization for magnetic monopoles.

• The nice thing about the path integral is that it generalizes immediately to quantum fields, and for that matter to all types (scalars, fermions, gauge fields).