2/13/09 Lecture 12 outline

General integrals

$$I_n(a) = \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+a)^n}$$

with n integer and $Im(a) > 0$ and k in Minkowski space. See

$$I_n = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1}}{da^{n-1}}I_1(a), \qquad I_1 = \frac{-i}{16\pi^2}\int_0^{\Lambda^2} du \frac{u-a+a}{u-a}$$

where we used the solid angle $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$, which is $2\pi^2$ for $D = 4$. Get

$$I_n(a) = i\left(16\pi^2(n-1)(n-2)a^{n-2}\right)^{-1} \qquad \text{for} \quad n \geq 3.$$

Special cases

$$I_1 = \frac{i}{16\pi^2}a\ln(-a) + \ldots,$$

$$I_2 = \frac{-i}{16\pi^2}\ln(-a) + \ldots,$$

where $\ldots$ are terms involving the regulator.

- Recall from last time: the 1-loop term in $\Gamma^{(2)}$ for $\lambda\phi^4$

$$\Pi'(p^2) = \tfrac{1}{2}\lambda\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2} + \text{more loops.}$$

Let's illustrate another, extremely popular, choice of regulator: dimensional regularization. Suppose that we had D instead of 4 dimensions. Compute by analytic continuation in D. Then take $D = 4 - \epsilon$, and take $\epsilon \to 0$. By going slightly below 4 dimensions, we improve the UV behavior (make the theory weaker in the UV, though stronger in the IR).

So we write

$$I \equiv \int\frac{d^Dk_E}{(2\pi)^D}\frac{1}{k_E^2+m^2} = \frac{\Omega_{D-1}}{(2\pi)^D}\int_0^\infty u^{D-1}du\frac{1}{u^2+m^2}.$$

Again, $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit sphere S^{D-1}. Let $u^2 = m^2y$

$$I = \frac{m^{D-2}}{2^D\pi^{D/2}\Gamma(D/2)}\int_0^\infty\frac{y^{(D-2)/2}dy}{y+1}.$$

Now use $(y-1)^{-1} = \int_0^\infty dte^{-t(y-1)}$ and $\Gamma(z) = \int_0^\infty dte^{-t}t^{z-1}$ to get

$$I = \frac{m^{D-2}}{(4\pi)^{D/2}}\Gamma(1-\tfrac{1}{2}D).$$

This blows up for $D = 4$, because $\Gamma(1 - \frac{1}{2}D)$ has a pole there. Recall $\Gamma(z)$ has a simple pole at $z = 0$, and also at all negative integer values of z.

Recall that near $x = 0$,

$$\lim_{x \to 0} \Gamma(x) = \frac{1}{x} - \gamma + \mathcal{O}(x),$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. For $x = -n$, we can write a similar expression, which also follows from the above and $\Gamma(z + 1) = z\Gamma(z)$. This gives

$$\lim_{x \to -n} \Gamma(x) = \frac{(-1)^n}{n!}(\frac{1}{x+n} - \gamma + 1 + \ldots + \frac{1}{n} + \mathcal{O}(x + n).).$$

E.g. use $\Gamma(2 - D/2) = (1 - D/2)\Gamma(1 - D/2)$. Let $D = 4 - \epsilon$, then (dropping $\mathcal{O}(\epsilon)$,

$$\frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}} \Delta^{D/2-2} \to \frac{1}{(4\pi)^2} \left(\frac{2}{\epsilon} - \log \frac{\Delta}{4\pi} - \gamma \right).$$

We can apply this to evaluate $\Pi^{(1)}(p^2)$. One last thing: replace $\lambda_{old} = \lambda_{new} \mu^{4-D}$, where λ_{new} is dimensionless. Expanding around $D = 4$, we get

$$\Pi'(p^2)^{(1)} = -\frac{\lambda m^2}{32\pi^2} \left(\frac{2}{\epsilon} - \log \frac{m^2}{4\pi\mu^2} + 1 - \gamma \right).$$

- More useful integrals:

$$\int \frac{d^D k_E}{(2\pi)^D} \frac{1}{(k_E^2 + \Delta)^n} = \frac{1}{4\pi)^{D/2}} \frac{\Gamma(n - \frac{1}{2}D)}{\Gamma(n)} \Delta^{D/2-n}.$$

$$\int \frac{d^D k_E}{(2\pi)^D} \frac{k_E^2}{(k_E^2 + \Delta)^n} = \frac{1}{4\pi)^{D/2}} \frac{D}{2} \frac{\Gamma(n - \frac{1}{2}D - 1)}{\Gamma(n)} \Delta^{1+D/2-n}.$$

- Now consider $\tilde{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)$. There are three 1-loop diagrams, in the s, t, u channels. Recall $s = (p_1 + p_2)^2$, $t = (p_1 + p_3)^2$, $u = (p_1 + p_4)^2$, $s + t + u = 4m^2$. Get

$$\tilde{\Gamma}^{(4)} = -\lambda\hbar^{-1} + (-i\lambda)^2(F(s) + F(t) + F(u)) + O(\hbar),$$

where

$$F(p^2) = \tfrac{1}{2} i \int \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 - m^2} \frac{1}{(k+p)^2 - m^2}.$$

The $\frac{1}{2}$ is a symmetry factor. Evaluate using

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2}.$$

Aside: more generally, have

$$\prod_{j=1}^{n} A_j^{-\alpha_j} = \frac{\Gamma(\sum_j \alpha_j)}{\prod_j \Gamma(\alpha_j)} \int_0^1 dx_1 \dots \int_0^1 dx_n \delta(1 - \sum_j x_j) \frac{\prod_k x^{\alpha_k - 1}}{(\sum_i x_i A_i)^{\sum \alpha_j}}.$$

Get

$$F(p_E^2) = -\int \frac{d^4 k_E}{(2\pi)^4} \int_0^1 dx \frac{1}{(xk_E^2 + m^2 + (1-x)(k_E + p_E)^2)^2}.$$

The quantity in the denominator is $k_E^2 + (1-x)2k_E \cdot p_E + (1-x)p_E^2 + m^2 = (k_E + (1-x)p_E)^2 + p_E^2(1-x)x + m^2$, so

$$F(s_E) = -\int \frac{d^4 k_E}{(2\pi)^4} \int_0^1 dx \frac{1}{(k_E^2 + m^2 + x(1-x)s_E)^2}.$$

Where $s_E = p_E^2 = -s$. Evaluate the k integral using the dimreg integrals above. Expanding around $D = 4 - \epsilon$, this gives

$$F(s_E) = -\frac{1}{32\pi^2} \int_0^1 dx \left(\frac{2}{\epsilon} - \gamma + \log(4\pi) - \log(m^2 + x(1-x)s_E) \right).$$

So the one-loop contribution to $\tilde{\Gamma}^{(4)}$ is

$$\frac{\lambda^2}{32\pi^2} \left(\frac{2}{\epsilon} - \gamma + \log \frac{4\pi\mu^2}{m^2} - \int_0^1 dx \log(1 + x(1-x)\frac{s_E}{m^2}) \right) + (s \to t) + (s \to u).$$

The integral is evaluated using

$$\int_0^1 dx \log(1 + \frac{4}{a} x(1-x)) = -2 + \sqrt{1+a} \log \left(\frac{\sqrt{1+a}+1}{\sqrt{1+a}-1} \right) \qquad a > 0.$$

• Next time: Renormalization. The input to the functional integral is the "bare" lagrangian. It is not physically observable, because we observe quantities like mass, charge, etc. with all the quantum corrections included. Write the largrangian for the bare fields as:

$$\mathcal{L}_B = \tfrac{1}{2}\partial_\mu \phi_B \partial^\mu \phi_B - \tfrac{1}{2} m_B^2 \phi_B^2 - \lambda_B \frac{1}{4!} \phi_B^4.$$

The bare field is related to the physical one by $\phi_B \equiv Z_\phi^{1/2} \phi$. We can view this as

$$\mathcal{L}_B = \mathcal{L}_{phys} + \mathcal{L}_{c.t.}$$

where

$$\mathcal{L}_{phys} = \tfrac{1}{2}\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2} m^2 \phi^2 - \lambda \frac{1}{4!} \phi^4$$

involves the physical field, mass, coupling constant. What's left are the counterterms:

$$\mathcal{L}_{c.t.} = \tfrac{1}{2}(Z-1)\partial_\mu \phi \partial^\mu \phi - \tfrac{1}{2}(m_B^2 Z - m^2)\phi^2 - (\lambda_B Z^2 - \lambda)\frac{1}{4!}\phi^4.$$