2/12/09 Lecture 11 outline

• Let's consider the 1-loop term in $\tilde{\Gamma}^{(2)}$ for $\lambda\phi^4$. Get

$$-i\Pi'(p^2) = (-i\lambda)\tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2} + \text{more loops}.$$

Now rotate to Euclidean space, $d^4k = id^4k_E$,

$$\Pi'(p^2) = \tfrac{1}{2}\lambda\int\frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2} + \text{more loops}.$$

Recall expression $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit sphere S^{D-1}. For $D=4$, get $\Omega_3 = 2\pi^2$, so

$$\Pi'(p^2) = \frac{\lambda m^2}{32\pi^2}\int_0^{\Lambda^2/m^2}\frac{udu}{u+1} = \frac{\lambda m^2}{32\pi^2}\left(\frac{\Lambda^2}{m^2} - \log(1+\frac{\Lambda^2}{m^2})\right).$$

Here Λ is a UV momentum cutoff. Result is quadratically (and also log) divergent as $\Lambda\to\infty$. The subject of renormalization is the physical interpretation of these divergences. The first thing to do is to regulate them, as we did above with a momentum cutoff. There are other ways to regulate. How one regulates is physically irrelevant. The physics is in the renormalization interpretation of the regulated results, and at the end of the day the choice of regulator doesn't matter.

• Study more generally the degree of divergence of 1PI diagrams. Consider the general form of $\Gamma^{(n)}$:

$$\Gamma^{(n)} \sim \int\prod_{i=1}^{L}\frac{d^4k_i}{(2\pi)^4}\prod_{j=1}^{I}\frac{1}{l_j^2-m^2+i\epsilon}$$

For large k the integrand behaves as $\sim k^{4L-2I}$. Degree of UV divergence (superficially) is $D = 4L-2I = 2I-4V+1$ (recall that $L = I-V+1$). Suppose interaction is ϕ^p, then $pV = 2I+n$. E.g. for $\lambda\phi^4$, $p=4$, get $D = 4-n$. For $p=6$, write $4V_4+6V_6 = 2I+n$, get $D = 4-n+2V_6$. The V_4 vertex is renormalizable, the V_6 is not. For $\lambda\phi^4$, the UV divergent terms are $n=2,4$. Higher n diagrams only have sub-divergences, which will be accounted for by properly treating the $n=2$ and $n=4$ cases. Example of a $n=6$ diagram with a sub-divergence from the $n=2$ diagram. Contrast $\lambda_4\phi^4$ with a $\lambda_3\phi^3$ theory (super-renormalizable) and a $\lambda_6\phi^6$ theory (non-renormalizable).

• Dimensional analysis and understanding the degrees of divergence by power-counting. In $\hbar = c = 1$ units, dimensionful quantities can be written as $x\sim m^{[x]}$, which defines $[x]$, the mass dimension of x. In particular, in D space-time dimensions, we have

$[S] = 0$ and $[d^D x] = -D$, so $[\mathcal{L}] = D$ so scalars have $[\phi] = (D-2)/2$ and fermions have $[\psi] = (D-1)/2$. We see that a $\lambda_p \phi^p$ theory has $[\lambda_p] = D - p(D-2)/2$. In particular, for $D = 4$, get $[\lambda_p] = 4 - p$, showing why $p = 4$ is special, as compared with say $\lambda_3 \sim M$ and $\lambda_6 \sim M^{-2}$. Since $\Gamma^{(n)}$ has units of action, i.e. $\hbar$, it has $[\Gamma^{(n)}] = 0$. So a contribution with e.g. V_6 vertices has, on dimensional grounds, a factor of $(\lambda_6 E^2)^{V_6}$, where E is some energy scale. This reproduces the degree of UV divergence if we take $E \sim \Lambda \to \infty$. Discuss similar power counting for gravity, and for Fermi's 4-fermion weak-interaction vertex. Interpretation as low-energy effective theory with cutoff. "Non-renormalizable" theories are fine, and actually nice, in the IR, and just need some fixing up in the UV, but some UV completion.