2/5/09 Homework 3. Due Feb 12

1. As mentioned in lecture, the Coleman-Weinberg potential for $V_{int} = \frac{\lambda}{4!}\phi^4$ is

$$V_1(\overline{\phi}) = i\sum_{n=1}^{\infty}\frac{1}{2n}\int\frac{d^4k}{(2\pi)^4}\left(\lambda\frac{1}{k^2-m^2+i\epsilon}\frac{\overline{\phi}^2}{2}\right)^n$$
$$= \tfrac{1}{2}\int\frac{d^4k_E}{(2\pi)^4}\ln\left(1+\frac{\frac{1}{2}\lambda\overline{\phi}^2}{k_E^2+m^2}\right)$$

(S. Coleman and E. Weinberg.) Symmetry factors: $1/n!$ not all the way cancelled, because of Z_n rotation symmetry, and reflection, gives $1/2n$. At each vertex, can exchange external lines, so $1/4!$ not all the way cancelled, leads to $1/2$ for each vertex. Here you will derive the last expression above for $V_1(\overline{\phi})$ another way. Write

$$e^{iW[J]} = \int[d\phi]e^{i(S+\int J\phi)/\hbar}$$

and expand $\phi(x) = \bar{\phi} + \eta(x)$, treating $\bar{\phi}$ as a constant and imagining η to be a small fluctuation, and keeping only terms to order η^2. Do the Gaussian integral over η formally. Then use the relation given in class to convert $W[J]$ into $\Gamma[\bar{\phi}]$ to finally reproduce the above expression for $V_1(\overline{\phi})$. Hint: use $\ln\det B = \mathrm{Tr}\ln B$ for any operator B, so e.g. $\ln\det(\partial^2+m^2) = V_4\int\frac{d^4k}{(2\pi)^4}\ln(-k^2+m^2)$ (where V_4 is a spacetime box size which can be taken to infinity at the end of the day; it cancels anyway).

2. For a scalar field theory, with $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi)$, show that the EOM are satisfied, up to a contact term:

$$\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x_\mu}\langle T\phi(x)\phi(y)\rangle + \langle V'(\phi(x))\phi(y)\rangle = \alpha\delta(x-y).$$

To do this problem, consider the functional integral for $W[J]$, and use the invariance of the functional integral under a change of variables. The change of variables is $\phi\to\phi+\epsilon f(x)$, where $f(x)$ is an arbitrary function of x, and ϵ is an infinitesimal parameter (drop terms of order ϵ^2 and higher). Derive in this way the above result (and determine the coefficient α). (Don't worry about the Jacobian in the $[d\phi]$ integration measure - it doesn't contribute.) The source term for ϕ is J, and you can think of $f(x)$ as a source for the EOM.

3 This problem is **optional**. (It's very similar to problem 2.) Consider $\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi - \frac{\lambda}{4}\phi^2\phi^{*2}$, where ϕ is a complex scalar field. The lagrangian has a symmetry,

$\phi \to e^{ia}\phi$, so there is a corresponding conserved current $J^\mu(x)$, with $\partial_\mu J^\mu = 0$. Find $J^\mu(x)$, using the Noether method. Now, in analogy with the above problem, use the functional integral to show that the current is conserved in correlation functions, up to contact terms:

$$\frac{\partial}{\partial x}\langle T J^\mu(x)\phi(y)\rangle = \beta\delta(x-y).$$

Derive this similarly to problem 2 (and, again determine the constant β).