2/7/07 Lecture 9 outline

• Let's consider the 1-loop term in $\tilde{\Gamma}^{(2)}$ for $\lambda\phi^4$. Get

$$-i\Pi'(p^2) = (-i\lambda)\tfrac{1}{2}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2} + \text{more loops.}$$

Now rotate to Euclidean space, $d^4k = id^4k_E$,

$$\Pi'(p^2) = \tfrac{1}{2}\lambda\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2} + \text{more loops.}$$

Recall expression $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit sphere S^{D-1}. For $D=4$, get $\Omega_3 = 2\pi^2$, so

$$\Pi'(p^2) = \frac{\lambda m^2}{32\pi^2}\int_0^{\Lambda^2/m^2}\frac{udu}{u+1} = \frac{\lambda m^2}{32\pi^2}\left(\frac{\Lambda^2}{m^2} - \log(1+\frac{\Lambda^2}{m^2})\right).$$

Here Λ is a UV momentum cutoff. Result is quadratically (and also log) divergent as $\Lambda\to\infty$. The subject of renormalization is the physical interpretation of these divergences. The first thing to do is to regulate them, as we did above with a momentum cutoff. There are other ways to regulate. How one regulates is physically irrelevant. The physics is in the renormalization interpretation of the regulated results, and at the end of the day the choice of regulator doesn't matter.

• Study more generally the degree of divergence of 1PI diagrams. Consider the general form of $\Gamma^{(n)}$:

$$\Gamma^{(n)} \sim \int \prod_{i=1}^{L}\frac{d^4k_i}{(2\pi)^4}\prod_{j=1}^{I}\frac{1}{l_j^2-m^2+i\epsilon}$$

For large k the integrand behaves as $\sim k^{4L-2I}$. Degree of UV divergence (superficially) is $D = 4L-2I = 2I-4V+1$ (recall that $L = I-V+1$). Suppose interaction is ϕ^p, then $pV = 2I+n$. E.g. for $\lambda\phi^4$, $p=4$, get $D = 4-n$. For $p=6$, write $4V_4+6V_6 = 2I+n$, get $D = 4-n+2V_6$. The V_4 vertex is renormalizable, the V_6 is not. This is apparent from powercounting of the dimension of the interaction. For $\lambda\phi^4$, the UV divergent terms are $n=2,4$. Higher n diagrams only have sub-divergences, which will be accounted for by properly treating the $n=2$ and $n=4$ cases.

• Consider again the 1-loop term in $\Gamma^{(2)}$ for $\lambda\phi^4$. Get

$$\Pi'(p^2) = \tfrac{1}{2}\lambda\int \frac{d^4k_E}{(2\pi)^4}\frac{1}{k_E^2+m^2} + \text{more loops.}$$

Let's illustrate another, extremely popular, choice of regulator: dimensional regularization. Suppose that we had D instead of 4 dimensions, then write

$$I \equiv \int \frac{d^D k_E}{(2\pi)^D} \frac{1}{k_E^2 + m^2} = \frac{\Omega_{D-1}}{(2\pi)^D} \int_0^\infty u^{D-1} du \frac{1}{u^2 + m^2}.$$

Again, $\Omega_{D-1} = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit sphere S^{D-1}. Let $u^2 = m^2 y$

$$I = \frac{m^{D-2}}{2^D \pi^{D/2} \Gamma(D/2)} \int_0^\infty \frac{y^{(D-2)/2} dy}{y+1}.$$

Now use $(y-1)^{-1} = \int_0^\infty dt e^{-t(y-1)}$ and $\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1}$ to get

$$I = \frac{m^{D-2}}{(4\pi)^{D/2}} \Gamma(1 - \tfrac{1}{2} D).$$

This blows up for $D = 4$, because $\Gamma(1 - \frac{1}{2} D)$ has a pole there. Recall $\Gamma(z)$ has a simple pole at $z = 0$, and also at all negative integer values of z.