1/31/07 Lecture 7 outline

• Last time: 2-point function, via summing geometric series,

$$D(p)=\frac{i}{\tilde{\Gamma}^{(2)}}=\frac{i}{p^2-m^2-\Pi'(p^2)}.$$

$-i\Pi'$ is computed from the 1PI diagrams. It starts at one-loop. $\Pi'(p^2)$ is called the self-energy, like momentum dependent mass term. The special definition of $\tilde{\Gamma}^{(2)}$ is because $D(p)=i/\tilde{\Gamma}^{(2)}$ will be nice, and allow extending to higher point functions. The point of the 1PI diagrams is that the quantum loop corrections are simply obtained by replacing the vertices with the 1PI greens functions! Indeed, Draw pictures for $n=2,4,6$ point functions. Obtain the full $W[J]$ via tree-graphs assembled from the 1PI building blocks.
• Let's write factors of $\hbar$: $\tilde{\Gamma}^{(2)}=\hbar^{-1}(p^2-m^2)-\Pi'(p^2)$, where $\Pi'(p^2)$ has terms of order $\hbar^0$ (1-loop), and higher.

• Note that there are are no tree level IPI diagrams for $\tilde{\Gamma}^{(n)}$ except for $n=4$ in $\lambda\phi^4$, so $\tilde{\Gamma}^{(n)}=-\lambda\hbar^{-1}\delta_{n,4}+\mathcal{O}(\hbar^0)+\dots$. At order $\hbar^0$, i.e. 1-loop, note that there are terms for all even n. There can not be terms for odd n, because of the $\phi\to-\phi$ symmetry.

• There is also a generating function for the 1PI green's functions:

$$\Gamma[\phi]=\sum_{n=1}^{\infty}\frac{1}{n!}\int d^4x_1\dots d^4x_n\Gamma^{(n)}(x_1,\dots x_n)\phi(x_1)\dots\phi(x_n).$$

This quantity is called the effective action. Find that

$$\Gamma[\phi]=\frac{1}{\hbar}\left(S[\phi]+\mathcal{O}(\hbar)\right).$$

E.g. in $\lambda\phi^4$, $\Gamma[\phi]=\hbar^{-1}\int d^4x[\frac{1}{2}\phi(-\partial^2-m^2)\phi-\frac{1}{4!}\lambda\phi^4]$+(quantum corrections). The quantum corrections are e.g. corrections to the mass from $m^2\to m^2+\hbar\Pi'(p^2)$, a correction to λ at order $\hbar$, and higher powers of ϕ at order $\hbar^{-1}(\hbar^L)$ for $L\geq 1$.

• Connecting $\Gamma[\phi]$ and $W[J]$. Introduce a (to count loops, formally take $a\to 0$):

$$e^{iW[J,a]}\equiv N\int[d\phi]e^{i(\Gamma[\phi]+\int d^4xJ\phi)/a}.$$

Then LHS=$\exp(i(W[J]+O(a))/a)$. Evaluate RHS by stationary phase:

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)}=-J(x)\qquad\text{for}\quad\phi=\overline{\phi}(x),$$

which is some functional of J. So the RHS is

$$Ne^{i(\Gamma[\overline{\phi}]+\int d^4x J\overline{\phi}+\mathcal{O}(\sqrt{a}))}.$$

Conclude

$$W[J] = \Gamma[\overline{\phi}] + \int d^4x J(x)\overline{\phi}(x).$$

This is a Legendre transform. Like $F = E - TS$ in Stat Mech. There is also the inverse transform:

$$\Gamma[\overline{\phi}] = W[J] - \int d^4x J(x)\overline{\phi}(x).$$

$\overline{\phi}(x)$ can be interpreted as the average of $\phi(x)$ in the presence of the source; sometimes called classical field:

$$\overline{\phi}(x) = \frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\phi(x)|0\rangle_J}{\langle 0|0\rangle_J}.$$

The functional derivatives of $\Gamma[\overline{\phi}]$, upon setting $\overline{\phi} = 0$, give $\Gamma^{(n)}(x_1, \dots x_n)$. In particular,

$$\frac{\delta\Gamma[\phi_c]}{\delta\overline{\phi}(x)}\Big|_{\overline{\phi}=0} = \Gamma^{(1)}(x) = 0.$$