1/10/07 Lecture 3 outline

• At the end last time, considered the path integral

$$Z_0 = \int [d\phi] \exp(\frac{i}{\hbar} S)$$

e.g. for the free Klein-Gordon field, $S = \frac{1}{2}\int d^4x\phi(x)(-\partial^2 - m^2)\phi(x)$, we said $Z_0 = \mathrm{const}(\det(-\partial^2 - \mathrm{m}^2))^{-1/2}$.

• Consider again the ordinary integral

$$\int_{-\infty}^{\infty} d\phi \exp(ia\phi^2) = \sqrt{\frac{i\pi}{a}}.$$

where we analytically continued from the case of an ordinary gaussian integral. Think of a as being complex. Then the integral converges for $\mathrm{Im}(a) > 0$, since then it's damped. To justify the above, for real a, we need the integral to be slightly damped, not just purely oscillating. To get this, take $a \to a + i\epsilon$, with $\epsilon > 0$, and then take $\epsilon \to 0^+$.

Now we should do the same thing for our functional integral above. I.e. we should take $S = \frac{1}{2}\int d^4x\phi(x)(-\partial^2 - m^2 + i\epsilon)\phi(x)$, with $\epsilon > 0$, and then $\epsilon \to 0^+$. Note that the operator is $B \sim -\partial^2 - m^2 + i\epsilon$, which in momentum space is $p^2 - m^2 + i\epsilon$. Looks familiar! The Feynman $i\epsilon$ prescription, which you understood last quarter as needed to give correct causal structure of greens functions, here comes simply from ensuring that the integrals converge! This is why the path integral automatically gives the time ordering of the products.

• Now discuss generating functions. First consider ordinary (non-functional) gaussian integrals. We'd like to evaluate integrals like

$$\prod_{i=1}^{N} \int d\phi_i f(\phi_i) \exp(-(\phi, B\phi))$$

for functions, like products of the ϕ_i. We can do this by computing a generating function:

$$\prod_{i=1}^{N} \int d\phi_i f(\phi_i) \exp(-B_{ij}\phi_i\phi_j) = f(\frac{\partial}{\partial J_i}) Z(J_i)\big|_{J_i=0}$$

Where

$$Z(J_i) \equiv \prod_{i=1}^{N} \int d\phi_i \exp(-B_{ij}\phi_i\phi_i + J_i\phi_i)$$

Evaluate via completing the square: the exponent is $-(\phi, B\phi) + (J, \phi) = -(\phi', B\phi') + \frac{1}{4}(J, B^{-1}J)$, where $\phi' = \phi - \frac{1}{2}B^{-1}J$. So

$$Z(J_i) = \prod_{i=1}^{N} \int d\phi_i \exp(-B_{ij}\phi_i\phi_j + J_i\phi_i) = \pi^{N/2}(\det B)^{-1/2} \exp(B_{ij}^{-1} J_i J_j / 4)$$

Similarly, we can compute field theory Green's functions via the functional integral analog of the above. The generating functional is

$$Z[J(x)] = \int [d\phi] \exp(i \int d^4x[\mathcal{L} + J(x)\phi(x)]).$$

This is a functional: input function $J(x)$ and it outputs a number. Use it to compute

$$\langle 0|T \prod_{i=1}^{n} \phi(x_i)|0\rangle / \langle 0|0\rangle = Z[J]^{-1} \prod_{j=1}^{n} \left(-i \frac{\delta}{\delta J(x_i)}\right) Z[J]\big|_{J=0}.$$

E.g. free field theory, in analogy with the above, we have $B = (-i/2\hbar)(-\partial^2 - m^2 + i\epsilon)$, so $B^{-1} = 2i\hbar(-\partial^2 - m^2 + i\epsilon)^{-1}$. We then get for the generating functional

$$Z[J] = Z_0 \exp(-\tfrac{1}{2}\hbar \int d^4x d^4y J(x) D_F(x-y) J(y)),$$

with

$$D_F(x-y) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ik(x-y)}}{k^2 - m^2 + i\epsilon},$$

D_F gives $i(-\partial^2 - m^2 + i\epsilon)^{-1}$.

Can use this generating function to compute free field time ordered products.. it reproduces Wick's theorem.

More generally, including interactions, the path integral reproduces the results of using Dyson and Wick's theorem, or alternatively of Feynman diagrams.

To see this, notice that

$$\int [d\phi] \exp(\frac{i}{\hbar}[S_{free} + S_{int}[\phi] + \hbar \int d^4x J\phi]) = \exp[\frac{i}{\hbar} S_{int}[-i\frac{\delta}{\delta J}]) Z_{free}[J].$$

So

$$Z[J] = N \exp[\frac{i}{\hbar} S_{int}[-i\frac{\delta}{\delta J}]) Z_{free}[J],$$

where N is an irrelevant normalization factor (independent of J). Correspondingly, the green's functions are given by

$$G^{(n)}(x_1 \ldots x_n) = \frac{\int [d\phi] \phi(x_1) \ldots \phi(x_n) \exp(\frac{i}{\hbar} S_I[\phi]) \exp[\frac{i}{\hbar} S_{free}]}{\int [d\phi] \exp(\frac{i}{\hbar} S_I[\phi]) \exp[\frac{i}{\hbar} S_{free}]}.$$

The denominator cancels off the vacuum bubble diagrams, which don't depend specifically on the Green's function. Next time: show how this relates to Feynman diagrams, including interactions!