3/16/07 Lecture 20 outline

• Bare and renormalized fields, and counterterms. $\psi_B = Z_2^{1/2}\psi_R$, $A_B^\mu = Z_3^{1/2}A_R^\mu$, $e_B Z_2 Z_3^{1/2} = e_R Z_1$. $\mathcal{L}_B = \mathcal{L}_R + \mathcal{L}_{c.t.}$.

$$\mathcal{L}_R = -\frac{1}{4}F_{R\mu\nu}F_R^{\mu\nu} + \bar{\psi}_R(i\partial\!\!\!/ - e_R A\!\!\!/_R - m_R)\psi_R,$$

$$\mathcal{L}_{ct} = -\frac{1}{4}\delta_3(F_{R\mu\nu})^2 + \bar{\psi}_R(i\delta_2\partial\!\!\!/ - \delta_1 e_R A\!\!\!/_R - \delta_m)\psi_R.$$

Where $\delta_1 = Z_1 - 1$, $\delta_2 = Z_2 - 1$, $\delta_3 = Z_3 - 1$, and $\delta_m = Z_2 m_0 - m$. We have

$$e_B Z_2 Z_3^{1/2} = e_R Z_1,$$

where the Z_1 will cancel the Z_1^{-1} in $\Gamma^\mu(q^2 = 0) = Z_1^{-1}\gamma^\mu$.

• Gauge invariance requires $Z_1 = Z_2$, since then $\delta_1 = \delta_2$ and the counterterm pieces have the same gauge invariance. Sure enough, direct calculation shows $Z_1 = Z_2$ (to all orders in perturbation, theory, and exactly)! So $e_R = \sqrt{Z_3}e_0 = e_{phys}$. Shows that renormalized charge is same for all species (e.g. electron and muon and anti-proton all have exactly the same effective charge).

• Note that $\Delta(eA) = 1$, gives $\Delta(e) = 2 - \frac{1}{2}D = \epsilon/2$, so

$$\alpha_B = \mu^\epsilon Z_3^{-1}\alpha_R.$$

Take $d/d\ln\mu$ and use $d\alpha_B/d\ln\mu = 0$ to get (just as we did for $\lambda\phi^4$)

$$0 = \epsilon\alpha Z_3^{-1} + \beta(\alpha,\epsilon)Z_3^{-1} + \beta(\alpha,\epsilon)\alpha\frac{d}{d\alpha}Z_3^{-1}.$$

where $\beta(\alpha,\epsilon) = d\alpha/d\ln\mu$. To have a smooth $\epsilon \to 0$ limit, we need $\beta(\alpha,\epsilon) = -\epsilon\alpha + \beta(\alpha)$, and then

$$\beta(\alpha) = \alpha^2\frac{da_1}{d\lambda}.$$

• Now let us compute a_1. $Z_3^{-1} = 1 - \Pi(0)$, so a_1 is minus the coefficient of the $1/\epsilon$ pole in $\Pi(0)$. Let us compute it.

• Loop correction to photon propagator, from virtual electron/positron loop:

$$i\Pi_2^{\mu\nu}(p) = i(p^2 g_{\mu\nu} - p^\mu p^\nu)\Pi(p^2).$$

$Z_3 = (1-\Pi(0))^{-1}$. To 1-loop, we take $\delta_3 = \Pi(0)$. To one-loop, find (See Peskin p. 251 for details)

$$\Pi_2^{\mu\nu}(p) = (p^2 g^{\mu\nu} - p^\mu p^\nu)(-8e^2/(4\pi)^{D/2}) \int_0^1 dx x(1-x)\Gamma(2-\tfrac{1}{2}D)/\Delta^{2-\frac{1}{2}D},$$

where $\Delta = m_e^2 - x(1-x)p^2$. We only care about the ϵ^{-1} term. Result to one loop:

$$\Pi(0) = -\frac{\alpha}{\pi}\epsilon^{-1}\frac{2}{3} + \text{finite}.$$

in MS, choose δ_3 to cancel the $1/\epsilon$ term only, so $\delta_3 = -\frac{\alpha}{\pi}\epsilon^{-1}\frac{2}{3}+$ higher loop. Finally, we have $Z_3^{-1} = 1 + \sum_k a_k \epsilon^{-k}$, with $a_1 = 2\alpha^2/3\pi$ to one loop.

This then gives

$$\beta(\alpha) = \frac{d\alpha}{d\ln\mu} = \alpha^2 \frac{da_1}{d\lambda} = \frac{2\alpha^2}{3\pi} + \text{higher loops}.$$

• The beta function is positive. Qualitatively similar to $\lambda\phi^4$: the theory is not asymptotically free. Integrate 1-loop beta function:

$$\alpha^{-1}(\mu) = -\frac{2}{3\pi}\ln(\frac{\mu}{\Lambda}).$$

Makes sense only for $\mu < \Lambda$, i.e. in the IR. Λ is a UV cutoff. Get $\alpha \to \infty$ as $\mu \to \Lambda$; this is the Landau pole. Looks bad, but we'll see the the energy scale where it blows up is so fantastically large that we don't need to worry (something new should fix it in the UV, e.g. grand unification can do the job). It does not run to zero in the IR, because there are no massless charged particles. It runs toward zero until it gets to the energy scale of the lightest charged particle, $m_e = 0.5MeV$, and then it stops running. So $137 = \frac{3}{3\pi}\ln(\Lambda/m_e)$. Gives $\Lambda = m_e \exp(137\pi)$, which too huge to worry about the apparent Landau pole there. (Other charged particles will bring the scale of Λ down to $\Lambda = m_e \exp(137\pi/N_f)$ where N_f is the effective number of charged particles, but it's still huge.)

Picture of vacuum polarization, of electron-positron pairs.

• QED vs QED. In QED, we have gauge invariance $\psi \to e^{ief(x)}\psi$, local $U(1)$ transformations. Generalize to local $SU(N_c)$ gauge transformations: $\psi \to \exp(igT^a f_a(x))\psi$, where T^a are traceless, Hermitian $N_c \times N_c$ matrices ($a = 1 \ldots N_c^2 - 1$), and ψ is a N_c column vector. Gauge conserved color charge. Need covariant derivatives, $\partial_\mu \to D_\mu = \partial_\mu + igA_\mu^a T^a$, i.e. introduce gauge fields, "gluons". The T_a matrices do not commute, $[T^a, T^b] = if_{abc}T^c$:

the group is “non-Abelian.” The effect of this is that the A_μ^a kinetic terms are more complicated. The physics of this is that the gluons carry color charge (unlike the photon, which carries no electric charge). Get 3 and 4 gluon interaction diagrams. Added contributions to 1-loop correction to gluon propagator. Get finally

$$\beta(\alpha) = \frac{\alpha^2}{6\pi}\left(-11N_c + 2N_f\right).$$

The flavors contribute positively, as in QED. But the colors contribute negatively: they anti-screen charges! So the beta function can be negative, if $11N_c > 2N_f$. This is asymptotic freedom. Integrating the 1-loop result gives

$$\alpha(\mu)^{-1} = \frac{(11N_c - 2N_f)}{6\pi}\ln(\frac{\mu}{\Lambda}).$$

To have $\alpha > 0$, we need $\mu > \Lambda$ (opposite from QED). Note $\alpha(\mu \to \infty) \to 0$, weak in UV = asymptotic freedom. Explains successes of parton model (quarks) for high energy scattering. For QCD, $N_c = 3$, and $N_f = 6$. For energies below the top and bottom mass, use $N_f^{eff} = 4$. Observe e.g. $\alpha(100GeV) \sim 0.1$, so $\Lambda \sim 200MeV$.

On the other hand, $\alpha \to \infty$ for $\mu \to \Lambda$: forces are strong in IR, below scale Λ. Can explain confinement of quarks (there is a million dollar prize, waiting to be collected, if you prove it in detail)!